A BENCHMARK TABLE FOR SIGNIFICANCE TEST IN THE CLASS OF ADAPTIVE REGRESSION MODEL

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We derive the precise analytic limit of the variance estimator of g, the concentrated ML estimator of γ_o in the adaptive regression model, and show that the limit and the original estimator generate vitually identical estimates of the variance of g and the corresponding significance test statistics. Then, based on the limit variance estimator we generate a table for the significance test of g in the adaptive regression model. The table may well be used for the same purpose even in the generalized model where g is extremely robust with respect to alternative specifications of the unknown covariance matrices of the parameter vector.

1. INTRODUCTION

Cooley/Prescott introduced the adaptive regression model [1], and later generalized it [4]. The generalized model henceforth referred to in short as C/P model may be written as:

$$y_t = x_t'\beta_t \qquad (t=1,2,3,...,T) \tag{1}$$

where the first element of $x_t(kx1)$ is unity, and the time process of $\beta_t(kx1)$ is represented by

$$\beta_t = \beta^p{}_t + u_t \tag{2}$$
$$\beta^p{}_t = \beta^p{}_{t-1} + w_t \quad .$$

u_t and w_t are assumed to be characterized by

$$u_t \sim N(0_{kx1}, \sigma_u^2\, \Sigma_u) \quad ;$$

$$w_t \sim N(0_{kx1}, \sigma_w^2\, \Sigma_w) \;;$$

$$\sigma_u^2 = (1-\gamma)\sigma^2 \text{ and } \sigma_w^2 = \gamma\sigma^2 \quad (\, 0 \leq \gamma \leq 1 \,) \tag{3}$$

where $\Sigma_u(kxk)$ and $\Sigma_w(kxk)$ are known up to scale factors.

In this model, parameter γ plays a pivotal role. If $\gamma = 0$, then β_t is stable over time, and otherwise unstable or drifts. Hence, testing H_0: $\gamma_0 = 0$, γ_0 the true γ, is of the prime importance in application of C/P model.

A consistent estimation of γ_0 can be made by maximizing the log likelihood function concentrated on γ:

$$L(\gamma) = Constant - \frac{T}{2}\ln s^2(\gamma) - \frac{1}{T}\ln|\Omega| \tag{4}$$

where

$$s^2(\gamma) = \frac{1}{T}[y - XB_{T+1}]'\Omega^{-1}[y - XB_{T+1}] \;;$$

$$y = [y_1\; y_2\; \ldots\; y_T]' \;;$$

$$X = [x_1\; x_2\; \ldots\; x_T]' \;;$$

$$B_{T+1} = [X'\Omega^{-1}X]^{-1}X'\Omega^{-1}y \;;$$

$$\Omega = \Omega(\gamma) = \gamma Q + (1-\gamma)R$$

in which $(i,j)th$ elements of Q and R are defined, respectively, as

$$q_{ij} = (x_i'\Sigma_w x_j)\cdot \min(T-i+1, T-j+1) \;;$$

$$r_{ij} = \delta_{ij}(x_i'\Sigma_u x_j) \qquad (i,j = 1,2,3,\ldots,T)$$

where δ_{ij} is Kronecker delta.

It is shown [4] that g, γ maximizing (4), is a consistent estimator of γ_0, and has the asymptotic variance of

$$V_T(\gamma_0) = \frac{2}{\frac{1}{T}\Sigma\frac{(d_i-1)^2}{[(d_i-1)\gamma_0+1]^2} - \left[\frac{1}{T}\Sigma\frac{d_i-1}{(d_i-1)\gamma_0+1}\right]^2} \tag{5}$$

where d_i is the *i-th* eigenvalue of a *TxT* square matrix $P = R^{-1/2}QR^{-1/2}$ whose *(i,j)th* element can be written as

$$p_{ij} = \left(x_i{}'\Sigma_w x_j/\sqrt{x_i{}'\Sigma_u x_i}\sqrt{x_j{}'\Sigma_u x_j}\right)\cdot\min(T-i+1, T-j+1) \quad . \tag{6}$$

Since asymptotically

$$\frac{\sqrt{T}(g-\gamma_0)}{\sqrt{V_T(g)}} \sim N(0,1) \ , \tag{7}$$

to test H_0: $\gamma_0 = 0$ Cooley/Prescott suggest to use

$$z_a = \frac{\sqrt{T}g}{\sqrt{V_T(g)}} \tag{8}$$

which is asymptotically standard normal under the null hypothesis.

2. THE ADAPTIVE REGRESSION MODEL AS A SPECIAL CASE

In the case of adaptive regression model where only intercept is subject to transitory and permanent variation, p_{ij} in (6) is reduced to $p_{ij} = \min(T-i+1, T-j+1)$. Consequently, d_i defined in (5) can be expressed [18, pp.272-273] as

$$d_i = \frac{1}{2+2\cos^2(2\phi)} = \frac{1}{4\cos^2\phi} \tag{9}$$

where

$$\phi = \phi(i) = \frac{\pi(T-i+1)}{2T+1} ,$$

so that

$$\begin{aligned} \zeta(\gamma_0) &\equiv \lim_{T\to\infty} V_T(\gamma_0) \\ &= \frac{1}{4}\left[\gamma_0(4-3\gamma_0)\right]^{3/2}\left[\sqrt{4-3\gamma_0} + \sqrt{\gamma_0}\right]^2 \end{aligned} \tag{10}$$

as derived in Appendix A.

However, convergence of $V_T(g)$ to $\zeta(g)$ over $g = (0,1]$ as T increases is remarkably rapid to the extent that $\zeta(g)$ is almost identical to $V_T(g)$ even for $T = 20$, hardly an asymptotic sample size, as reported in Table 1. Hence, replacing $V_T(g)$ with $\zeta(g)$, we obtain an analytic expression:

$$\begin{aligned} \tilde{z} &= \frac{\sqrt{T}g}{\sqrt{\zeta(g)}} \\ &= 2\sqrt{T}g\left[\left(4g - 3g^2\right)^3\left(\sqrt{4-3g} + \sqrt{g}\right)^4\right]^{-1/4} \end{aligned} \tag{11}$$

which generates values virtually identical to those by z_a as reported also in Table 1. However, though drastically simpler than z_a, $\bar{z}$ still remains somewhat cumbersome. Hence, a table for critical values of g for different significance levels, respectively, may be a very useful reference in testing H_0: $\gamma_0 = 0$. Table 2 reports the null critical values of g for 10%, 5%, 1% significance levels for $20 \leq T \leq 100$. The critical values are computer generated solutions for g of $\bar{z} = z_{\alpha/2}$ where $z_{\alpha/2}$ denotes the critical value of the standard normal variate for the significance level of α.

3. GENERAL CASE AND CONCLUSION.

C/P model not only has been discussed in econometric text books [10,11,16], but has been applied in a number of empirical studies [2,3,5,6,7,9,12,13,14,15,17,18] in which the ML estimator g and the corresponding null test statistics have been found to be remarkably robust with respect to misspecifications of Σ_u end Σ_w. The robustness is somewhat well summarized in Cooley and DeCarno [5, p.12]:

> *... Extensive experiments were carried out with alternative specifications of Σ_u and Σ_w, and the parameter histories traced out with these alternative covariance specifications were all very similar, with extremely high correlations between both the values of the parameters at different base periods (see below) and changes in the parameter values from one base period to the next. Comparisons of Baysian posterior odds did not indicate the superiority of any particular specifications of the Σ matrices (Zeller, 1971, pp. 291-302). Thus, the analysis presented below is quite robust with respect to alternative Σ specifications. ...*

For this very reason of strong robustness with respect to Σ matrices, the critical values in Table 2 for the adaptive regression model proper may be used for C/P model in general in testing H_0: $\gamma_0 = 0$. It is even more justifiable in light of the fact that both Σ_u and Σ_w are unknown in practice.

TABLE 1

g	$V_T(g)$	$\zeta(g)$	τ_1	z_a	$\bar{z}$	τ_2
.05	.09919	.10087	.016992	.70998	.70403	.008389
.10	.27767	.28225	.016524	.84869	.84176	.008191
.15	.49068	.50121	.014463	.95436	.94753	.007154
.20	.72673	.73586	.012570	1.04920	1.04267	.006226
.25	.96054	.97090	.010786	1.14076	1.13466	.005350
.30	1.18406	1.19478	.009057	1.23296	1.22742	.004498
.35	1.38837	1.39857	.007347	1.32840	1.32355	.003653
.40	1.56662	1.57544	.005629	1.42920	1.42520	.002803
.45	1.71361	1.72026	.003881	1.53735	1.53437	.001935
.50	1.82561	1.82941	.002080	1.65494	1.65322	.001039
.55	1.90022	1.90061	.000020	1.78433	1.78415	.000004
.60	1.93621	1.93280	.001764	1.92837	1.93007	.000090
.65	1.93349	1.92603	.003858	2.09053	2.09458	.001935
.70	1.89300	1.88144	.006108	2.27529	2.28227	.003068
.75	1.81665	1.80111	.008551	2.48852	2.49923	.004303
.80	1.70728	1.68809	.011239	2.73812	2.75364	.005667
.85	1.56863	1.54630	.014239	3.03510	3.05694	.007197
.90	1.40529	1.38050	.017643	3.39527	3.42562	.008940
.95	1.22267	1.19628	.021581	3.84334	3.88438	.010968
1.00	1.02600	1.00000	.026250	4.41305	4.47214	.013389

T = 20; $\tau_1 \equiv |\zeta(g) - V_T(g)|/V_T(g)$; $\tau_2 \equiv |\bar{z} - z_a|/z_a$

TABLE 2

Upper 10%, 5%, and 1% null critical values of g

T	10%	5%	1%	T	10%	5%	1%
21	.478	.595	.752	61	.146	.243	.422
22	.463	.580	.739	62	.142	.238	.416
23	.448	.566	.728	63	.138	.233	.411
24	.433	.552	.716	64	.135	.229	.405
25	.420	.539	.704	65	.131	.224	.400
26	.406	.526	.693	66	.128	.220	.395
27	.394	.513	.682	67	.125	.215	.390
28	.381	.501	.672	68	.122	.211	.385
29	.369	.490	.661	69	.119	.207	.380
30	.358	.478	.651	70	.116	.203	.375
31	.347	.467	.641	71	.113	.199	.370
32	.336	.457	.632	72	.110	.195	.366
33	.326	.446	.622	73	.107	.191	.361
34	.316	.436	.613	74	.105	.187	.356
35	.307	.426	.604	75	.102	.184	.352
36	.297	.417	.595	76	.100	.180	.347
37	.288	.407	.587	77	.097	.177	.343
38	.280	.399	.578	78	.095	.173	.339
39	.272	.390	.570	79	.093	.170	.334
40	.264	.381	.562	80	.091	.167	.330
41	.256	.373	.554	81	.089	.164	.326
42	.248	.365	.546	82	.087	.160	.322
43	.241	.357	.539	83	.085	.157	.318
44	.234	.349	.531	84	.083	.155	.314
45	.227	.342	.524	85	.081	.152	.310
46	.221	.334	.516	86	.079	.149	.306
47	.215	.327	.422	87	.078	.146	.303
48	.209	.320	.502	88	.076	.143	.299
49	.203	.313	.495	89	.074	.141	.295
50	.197	.306	.489	90	.073	.138	.292
51	.192	.300	.482	91	.071	.136	.288
52	.186	.294	.476	92	.070	.133	.284
53	.181	.288	.469	93	.068	.131	.281
54	.176	.281	.463	94	.067	.129	.278
55	.171	.276	.457	95	.065	.126	.274
56	.167	.270	.451	96	.064	.124	.271
57	.162	.264	.445	97	.063	.122	.268
58	.158	.259	.439	98	.061	.120	.264
59	.154	.254	.433	99	.060	.118	.261
60	.150	.248	.427	100	.059	.116	.258

APPENDIX A

Using Equations 858.540 and 858.542 in Dwight [2, p.219], we can show that for any positive constant a

$$\lambda(a) \equiv \int_0^{\pi/2} \frac{1}{a^2\cos^2\phi + 1} d\phi = \frac{\pi}{2\sqrt{1 + a^2}} \quad ;$$

$$\mu(a) \equiv \int_0^{\pi/2} \frac{1}{\left[a^2\cos^2\phi + 1\right]^2} d\phi = \frac{\pi(2 + a^2)}{4(1 + a^2)^{3/2}} \quad . \tag{A.1}$$

Let

$$\Delta\phi \equiv \frac{\phi(1) - \phi(T)}{T} \quad ;$$

$$g(\phi) \equiv \frac{d_i - 1}{(d_i - 1)\gamma_0 + 1} \quad ;$$

$$h(\phi) \equiv \frac{(d_i - 1)^2}{\left[(d_i - 1)\gamma_0 + 1\right]^2} \tag{A.2}$$

where ϕ and d_i are as in (9).

Then,

$$\theta_1 \equiv \lim_{T\to\infty} \left[\frac{1}{\Delta\phi} \sum_{i=1}^{T} g(\phi)\ \Delta\phi\right] = \frac{2T}{\pi} \int_0^{\pi/2} g(\phi)\, d\phi \quad ;$$

$$\theta_2 \equiv \lim_{T\to\infty} \left[\frac{1}{\Delta\phi} \sum_{i=1}^{T} h(\phi)\ \Delta\phi\right] = \frac{2T}{\pi} \int_0^{\pi/2} h(\phi)\, d\phi \quad . \tag{A.3}$$

$g(\phi)$ and $h(\phi)$ from (A.2) substituting into the definite integrals in (A.3), θ_1 and θ_2 can be expressed in short in terms of λ(a) and

μ(a) defined in (A.1):

$$\theta_1 = \frac{2T}{\pi(1-\gamma_0)}\left[\frac{1}{\gamma_0}\lambda(a) - \frac{\pi}{2}\right] \quad ;$$

$$\theta_2 = \frac{2T}{\pi(1-\gamma_0)^2}\left[\frac{\pi}{2} - \frac{2}{\gamma_0}\lambda(a) + \frac{1}{\gamma_0^2}\mu(a)\right] \tag{A.4}$$

where $a = 2\sqrt{(1-\gamma_0)/\gamma_0}$.

Since without proof

$$\zeta(\gamma_0) = \frac{2}{\frac{1}{T}\theta_2 - \left[\frac{1}{T}\theta_1\right]^2} \quad , \tag{A.5}$$

θ_1 and θ_2 from (A.4) substituting into (A.5) results in

$$\zeta(\gamma_0) = \frac{1}{4}\left[\gamma_0(4-3\gamma_0)\right]^{3/2}\left[\sqrt{4-3\gamma_0} + \sqrt{\gamma_0}\right]^2 \tag{A.6}$$

after some algebraic manipulations.

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