A KIT OF RESULTS FOR SAMPLED AND TEMPORALLY AGGREGATED MODELS

by

Luigi Ermini

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(*) Address for correspondence:

2424 Maile Way
Department of Economics
University of Hawaii
Honolulu, HI 96822

Tel (808) 956-8590
Fax (808) 956-4347

Abstract

Interest in the effect of sampling and temporal aggregation on empirical results in macroeconomics and finance is growing. While the effects on the order of **ARIMA** representations are well known in the literature, the effects on model parameters are not, with a few exceptions. This paper presents general expressions for the effects on parameter values that can be useful for researchers in this area. Applications to **IMA**(1,2) processes and to the cross-correlations of two **IMA**(1,1) processes are illustrated.

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A Kit of Results for Sampled and Temporally Aggregated Models

1. Introduction

A number of papers have recently appeared in the literature analyzing the effects of sampling and temporal aggregation on empirical results in macroeconomics and finance. See, among others, Grossman, Melino and Shiller [1987], Naik and Ronn [1988], Breeden, Gibbons and Litzenberger [1989], Longstaff [1989], and Ermini [1991, 1992a, 1993] for applications to the consumption-based capital asset pricing model; Christiano and Eichenbaum [1987], Ermini [1988, 1989, 1992c, 1992d], and Christiano, Eichenbaum and Marshall [1991] for applications to the consumption function and the permanent income hypothesis; Ermini [1989, 1992b], and Heaton [1989] for studies on the durability of non-durable goods; Rossana and Seater [1989] for studies on manufacturing-sector wages; Amemiya and Wu [1972], and Luetkepohl [1984] for the effects on forecasts.

Sampling and temporal aggregation affect both the parameters of the data generating mechanism and the order of its ARIMA representation. While the effects on the order of the ARIMA representation are well known in the literature (for a comprehensive treatment, see Weiss [1984]; for previous work, see, among others, Tiao [1972], Tiao and Wei [1976], and Wei [1981]), the effects on the values of the parameters are not, except for the case of random walks (Working [1960]) and IMA(1,1) processes (Ermini [1989]). The purpose of this paper is to provide practical closed-form expressions for the effects on parameter values that can be useful to researchers in this field, and to collect in a concise reference results found in scattered sources. As the focus here is limited to linear models, the analysis is confined to second moments; to increase readability, these results are gathered in the form of tables (section 2). Section 3 gives some examples of application, deriving the effects of sampling and temporal aggregation on the cross-correlations of two IMA(1,1) processes and on the parameters of IMA(1,2) processes. Some of these results are utilized in Ermini [1992a, 1993] to study the impact of habit formation and durability of non-durable goods on observed expenditures series under temporal aggregation. The results on cross-correlations are applied in Ermini [1991] to the bivariate system consumption-asset prices to propose a solution to the equity premium puzzle.

2. Effects of sampling and temporal aggregation on second moments

Let $\{X_s\}$ be the original series, generated at shorter intervals (of equal length), and let $\{\overline{X}_t\}$ be the observed series, generated from the former at longer intervals (again, of equal length) by sampling or by temporal aggregation. Let m be the sampling ratio, that is the ratio between the longer interval of observation and the shorter interval of generation; in what follows, m is an integer equal to or greater than one. Sampling generates the series $\overline{X}_t$ by selecting from the original series values distanced at m shorter periods. For example, using the index t to refer to the longer interval, $\overline{X}_t = X_{mt}$, and $\overline{X}_{t\text{-}1} = X_{mt\text{-}m}$. Temporal aggregation generates the series $\overline{X}_t$ by summing m adjacent values of X_t, and selecting from this aggregated series values distanced at m periods. For example, $\overline{X}_t = \sum_{j=0}^{m-1} X_{mt-j}$, and $\overline{X}_{t-k} = \sum_{j=0}^{m-1} X_{mt-mk-j}$. For this reason, temporal aggregation is more properly called non-overlapping temporal aggregation. The literature often considers also the case of time averaging. This case, however, will not be considered here, as a time averaged series simply corresponds to a temporally aggregated series divided by m, and thus the effects of time averaging are easily derived from the effects of temporal aggregation. In fact, the scaling factor $1/m^2$ introduced into second moments disappears when considering the effects on correlations.

Table 1 reports the relation between the observed series and the original series for various cases. As in macroeconomics models are often formulated in first-differences, table 1 explicitly reports these relations for first differences of the variables as well. The temporal aggregation operator, $T_m(B)$, is defined as $\sum_{j=0}^{m-1} B^j$, or equivalently as $(1\text{-}B^m)/(1\text{-}B)$, where B is the lag operator (e.g. such that $X_{t-k} = B^k X_t$). Thus, under sampling $\Delta\overline{X}_t = X_{mt} - X_{mt-m} = (1\text{-}B^m)X_{mt} = (1\text{-}B^m)/(1\text{-}B)\ \Delta X_{mt}$. Similarly under temporal aggregation, $\Delta\overline{X}_t = T_m(B)(X_{mt}\text{-}X_{mt-m}) = |T_m(B)|^2 \Delta X_{mt}$.

Table 2 reports the effects of sampling and temporal aggregation on unconditional autocovariances. From these expressions, we can derive some of the results about the *order* of ARIMA representations already obtained by Weiss [1984]. For example, as for a stationary series $\lim_{t\to\infty} R_x(m\tau) = 0$, it is seen that for m going to infinity sampling transforms any stationary series into white noise; similarly, sampling transforms any I(1) series into a random walk.

Table 3 reports the effects of sampling and temporal aggregation on unconditional bivariate cross-covariances, for three cases: both series are sampled; both series are temporally aggregated; one series is sampled and the other is temporally aggregated.

3. Examples of Application

(*a*) *Cross-correlations between two IMA(1,1) processes with orthogonal innovations.* Consider the bivariate system

$$\begin{aligned} \Delta X &= \alpha + \varepsilon_t + \gamma\,\varepsilon_{t-1} \\ \Delta Y &= \beta + \eta_t + \delta\,\eta_{t-1} \end{aligned} \tag{1}$$

with ε_t and η_t orthogonal innovations processes, e.g. such that $E(\varepsilon_t\,\eta_t) = \omega$, $E(\varepsilon_t\,\eta_s) = 0$ for all $t \neq s$. Then $R_{\Delta x\Delta y}(0) = \omega(1+\gamma\delta)$, $R_{\Delta x\Delta y}(1) = \gamma\omega$, $R_{\Delta x\Delta y}(\text{-}1) = \delta\omega$, and $R_{\Delta x\Delta y}(\tau) = 0$ for all $|\tau| > 1$.

Consider first the case in which both series are sampled. Then

$$R_{\Delta\bar{x}\Delta\bar{y}}(\tau) = mR_{\Delta x\Delta y}(m\tau) + \sum_{j=1}^{m-1} (m-j)\,[R_{\Delta x\Delta y}(m\tau+j) + R_{\Delta x\Delta y}(m\tau-j)]\,. \tag{2}$$

Some algebra shows that for the contemporaneous cross-covariance, $R_{\Delta\bar{x}\Delta\bar{y}}(0) = \omega(1+\gamma\delta)m + \omega\,\gamma(m-1) + \omega\,\delta(m-1)$, which for m sufficiently big reduces to $\omega(1+\gamma\delta+\gamma+\delta)m$. Also, it can be shown that the sampling ratio m does not affect the one-lagged cross-covariance, so that $R_{\Delta\bar{x}\Delta\bar{y}}(\pm 1) = R_{\Delta x\Delta y}(\pm 1)$. Finally, $R_{\Delta\bar{x}\Delta\bar{y}}(\tau) = 0$ for all $|\tau| > 1$. Note that as the one-lagged cross-covariance is independent on m, but the contemporaneous cross-covariance is linearly increasing with it, the one-lagged coefficient of cross-correlation, $\rho_{\Delta\bar{x}\Delta\bar{y}}(1) = R_{\Delta\bar{x}\Delta\bar{y}}(1)/R_{\Delta\bar{x}\Delta\bar{y}}(0)$, goes to zero as m goes to infinity. This result is explained by the limiting result anticipated in the previuous section whereby any I(1) series under sampling is transformed into a random walk as m goes to infinity.

Consider now the case in which both IMA(1,1) processes are temporally aggregated. Then

$$\begin{aligned} R_{\Delta\bar{x}\Delta\bar{y}}(\tau) = {} & m^2 R_{\Delta x\Delta y} + 2m \sum_{j=1}^{m-1} (m-j)\,[R_{\Delta x\Delta y}(m\tau+j) + R_{\Delta x\Delta y}(m\tau-j)] + \\ & \sum_{j=1}^{m-1}\sum_{i=1}^{m-1} (m-j)(m-i)\,[R_{\Delta x\Delta y}(m\tau+j+i) + R_{\Delta x\Delta y}(m\tau-j-i) + \\ & R_{\Delta x\Delta y}(m\tau+j-i) + R_{\Delta x\Delta y}(m\tau-j+i)] \end{aligned} \tag{3}$$

Consider the contemporaneous cross-covariance. The second term in the right-hand side of (3) is $2m(m-1)[R_{\Delta x\Delta y}(1) + R_{\Delta x\Delta y}(-1)]$. Regarding the third term, both $R_{\Delta x\Delta y}(j+i)$ and $R_{\Delta x\Delta y}(-j-i)$ are zero for all $j,i \geq 1$. For the remaining two terms, the double summation becomes $2R_{\Delta x\Delta y}(0) \sum_{j=1}^{m-1} (m-j)^2 + 2[R_{\Delta x\Delta y}(1) + R_{\Delta x\Delta y}(-1)] \sum_{j=2}^{m-1} (m-j)(m-j+1)$. With some algebra one gets $R_{\Delta\bar{x}\Delta\bar{y}}(0) = \omega[(1+\gamma\delta)m\,(2m^2+1) + (\gamma+\delta)2m\,(m^2-1)]/3$, which reduces for m sufficiently big to $\frac{2}{3}\omega(1+\gamma\delta+\gamma+\delta)m^3$. Similarly, after some tedious calculations, for the one-lagged cross-covariance one gets $R_{\Delta\bar{x}\Delta\bar{y}}(1) = \omega[(1+\gamma\delta)m\,(m^2-1) + \gamma(m^3+3m^2+2m) + \delta(m^3-3m^2+2m)]/6$, which for m sufficiently big reduces to $\omega(1+\gamma\delta+\gamma+\delta)m^3/6$. Note that for m sufficiently big the cross-covariogram becomes symmetric. Finally, it can be shown that $R_{\Delta\bar{x}\Delta\bar{y}}(\tau) = 0$ for all $|\tau| > 1$. Regarding the first-lag cross-correlation $\rho_{\Delta\bar{x}\Delta\bar{y}}(1)$, it is easy to see that its value goes to 0.25 as m goes to infinity. Remarkably, this limit is identical to the limit of the first-lag autocorrelation of an IMA(1,1) process under temporal aggregation (see below). These results are summarized in table 4.

(*b*) *Auto-correlations of an IMA(1,2) process*. Consider the process

$$\Delta X = \alpha + \varepsilon_t + \gamma_1\,\varepsilon_{t-1} + \gamma_2\,\varepsilon_{t-2}\,, \tag{4}$$

with ε_t an innovation process of variance σ_ε^2, and $R_{\Delta x}(0) = (1 + \gamma_1^2 + \gamma_2^2)\sigma_\varepsilon^2$, $R_{\Delta x}(\pm 1) = (\gamma_1 + \gamma_1\gamma_2)\sigma_\varepsilon^2$, $R_{\Delta x}(\pm 2) = \gamma_2\sigma_\varepsilon^2$, and $R_{\Delta x}(\tau) = 0$ for all $|\tau| > 2$. Consider first the case of sampling. From table 2

$$R_{\Delta\bar{x}}(\tau) = mR_{\Delta x}(m\tau) + \sum_{j=1}^{m-1} (m-j)\,[R_{\Delta x}(m\tau+j) + R_{\Delta x}(m\tau-j)]\,. \tag{5}$$

With some algebra, and introducing the auto-correlations of the original process, $\rho_{\Delta x}(\tau) = R_{\Delta x}(\tau)/R_{\Delta x}(0)$, one gets for the process variance $R_{\Delta\bar{x}}(0) = R_{\Delta x}(0)[m + 2(m-1)\rho_{\Delta x}(1) + 2(m-2)\rho_{\Delta x}(2)]$, which for a sufficiently big sampling ratio m reduces to $[1+2\rho_{\Delta x}(1)+2\rho_{\Delta x}(2)]m$. For the one-lagged auto-covariance, it can be shown that $R_{\Delta\bar{x}}(1) = R_{\Delta x}(1) + 2R_{\Delta x}(2)$, independent on the sampling ratio. Finally, one finds that $R_{\Delta x}(\tau) = 0$ for all $|\tau| > 1$. This result shows that sampling transforms an IMA(1,2) process into an IMA(1,1) process at any sampling ratio m. Furthermore, as the first-lag autocovariance is independent on m, but the variance is linearly increasing with it, the first-lag coefficient of auto-correlation, $\rho_{\Delta\bar{x}}(1) = R_{\Delta\bar{x}}(1)/R_{\Delta\bar{x}}(0)$, goes to zero as m goes to infinity, thus confirming the limiting result already anticipated in section 1 that sampling transforms any IMA(1,2) into a random walk. These results are summarized in table 4. By setting $\rho_{\Delta x}(2) = 0$ and $\rho_{\Delta x}(1) = 0$, the effects of sampling on IMA(1,1) processes and on random walks are also easily derived. Particularly, a random walk under sampling remains a random walk with a variance linearly increasing with the sampling ratio; an IMA(1,1) under sampling remains an IMA(1,1) for finite sampling ratios, and becomes a random walk for m going to infinity.

Consider now the case of temporally aggregating process (4). From table 2, with some algebra one obtains the expressions reported in table 5. Note that the lag-two auto-covariance is unaffected by the sampling ratio, and thus under temporal aggregation an IMA(1,2) process remains IMA(1,2) for finite sampling ratios, but becomes an IMA(1,1) as m goes to infinity. Regarding the limiting effects on the first-lag, it is easily seen that for a sufficiently big sampling ratio, we have $R_{\Delta\bar{x}}(0) = \frac{2}{3}m^3R_{\Delta x}(0)[1 + 2\rho_{\Delta x}(1) + 2\rho_{\Delta x}(2)]$, and $R_{\Delta\bar{x}}(1) = \frac{1}{6}m^3R_{\Delta x}(0)[1 + 2\rho_{\Delta x}(1) + 2\rho_{\Delta x}(2)]$, from which it is seen that the limit of the first-lag autocorrelation $\rho_{\Delta\bar{x}}(1) = 0.25$, result already obtained in Working [1960] and Ermini [1989] for random walks and IMA(1,1) processes respectively.

4. Conclusions

This paper has presented practical closed-form expressions for the effects of sampling and temporal aggregation on the parameters of ARIMA models. These expressions are applied to two popular models in macroeconomics and finance, the univariate IMA(1,2) process and the bivariate IMA(1,1) process. Regarding the former, it is found that an IMA(1,2) process under temporal aggregation remains an IMA(1,2) process for finite sampling ratios (with decreassing lag-two autocorrelation), and in the limit as the sampling ratio m goes to infinity, it becomes an IMA(1,1) with a first-lag autocorrelation of 0.25. Regarding the latter, it is found that the first-lag cross-correlation also has the 0.25 limit as the sampling ratiol goes to infinity.

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TABLE 1	RELATIONS BETWEEN THE ORIGINAL SERIES X_s AND THE SAMPLED OR TEMPORALLY AGGREGATED SERIES $\overline{X}_t$ (*)	
	level	first differences
sampling	$\overline{X}_t = X_{mt}$	$\Delta\overline{X}_t = T_m(B)\,\Delta X_{mt}$
temp. aggregation	$\overline{X}_t = T_m(B)\,X_{mt}$	$\Delta\overline{X}_t = \lvert T_m(B)\rvert^2\,\Delta X_{mt}$
$T_m(B) = \sum_{j=0}^{m-1} B^j = (1-B^m)/(1-B)$		
$\lvert T_m(B)\rvert^2 = m + \sum_{j=1}^{m-1}(m-j)(B^j + B^{-j})$		
$(\lvert T_m(B)\rvert^2)^2 = m^2 + 2m\sum_{j=1}^{m-1}(m-j)(B^j+B^{-j}) + \sum_{j=1}^{m-1}\sum_{i=1}^{m-1}(m-j)(m-i)(B^{i+j}+B^{-(i+j)}+B^{i-j}+B^{j-i})$		

TABLE 2	EFFECTS ON UNCONDITIONAL AUTOCOVARIANCES ($R_z(\tau) = E\,(Z_t-EZ_t)(Z_{t-\tau}-EZ_{t-\tau})$)	
	level	first differences
sampling	$R_{\overline{x}}(\tau) = R_x(m\tau)$	$R_{\Delta\overline{x}}(\tau) = \lvert T_m(B)\rvert^2\,R_{\Delta x}(m\tau)$
temp. aggregation	$R_{\overline{x}}(\tau) = \lvert T_m(B)\rvert^2\,R_x(m\tau)$	$R_{\Delta\overline{x}}(\tau) = (\lvert T_m(B)\rvert^2)^2\,R_{\Delta x}(m\tau)$

(*) The index t refers to the longer interval of the sampled or temporally aggregated series.

TABLE 3	EFFECTS ON UNCONDITIONAL CROSS-COVARIANCES $(R_{zw}(\tau) = E(Z_t - EZ_t)(W_{t-\tau} - EW_{t-\tau}))$	
	level	first differences
both sampled	$R_{\overline{x}\overline{y}}(\tau) = R_{xy}(m\tau)$	$R_{\Delta\overline{x}\Delta\overline{y}}(\tau) = \mid T_m(B) \mid^2 R_{\Delta x \Delta y}(m\tau)$
one sampled *one temp. aggr.*	$R_{\overline{x}\overline{y}}(\tau) = T_m(B)\, R_{xy}(m\tau)$	$R_{\Delta\overline{x}\Delta\overline{y}}(\tau) = T_m(B) \mid T_m(B) \mid^2 R_{\Delta x \Delta y}(m\tau)$
both temp. aggregated	$R_{\overline{x}\overline{y}}(\tau) = \mid T_m(B) \mid^2 R_{xy}(m\tau)$	$R_{\Delta\overline{x}\Delta\overline{y}}(\tau) = (\mid T_m(B) \mid^2)^2 R_{\Delta x \Delta y}(m\tau)$

TABLE 4	EFFECTS ON CROSS-COVARIANCES OF TWO IMA(1,1) PROCESSES WITH ORTHOGONAL INNOVATIONS (*)		
	any m	$m >> 1$	$\lim_{m \to \infty}$
(a) Both Series Sampled			
$R_{\Delta\bar{x}\Delta\bar{y}}(0)$	$m\omega(1+\gamma\delta+\gamma+\delta) - \omega(\gamma+\delta)$	$m\omega(1+\gamma\delta+\gamma+\delta)$	
$R_{\Delta\bar{x}\Delta\bar{y}}(1)$	$\omega\gamma$	$\omega\gamma$	
$R_{\Delta\bar{x}\Delta\bar{y}}(-1)$	$\omega\delta$	$\omega\delta$	
$\rho_{\Delta\bar{x}\Delta\bar{y}}(1)$			0
(b) Both Series Temporally Aggregated			
$R_{\Delta\bar{x}\Delta\bar{y}}(0)$	$\frac{1}{3}\omega[(1+\gamma\delta)m\,(2m^2+1) + (\gamma+\delta)2m\,(m^2-1)]$	$\frac{2}{3}\omega m^3(1+\gamma\delta+\gamma+\delta)$	
$R_{\Delta\bar{x}\Delta\bar{y}}(1)$	$\frac{1}{6}\omega[(1+\gamma\delta)m\,(m^2-1)+\gamma(m^3+3m^2+2m)+\delta(m^3-3m^2+2m)]$	$\frac{1}{6}\omega m^3(1+\gamma\delta+\gamma+\delta)$	
$R_{\Delta\bar{x}\Delta\bar{y}}(-1)$	$\frac{1}{6}\omega[(1+\gamma\delta)m\,(m^2-1)+\delta(m^3+3m^2+2m)+\gamma(m^3-3m^2+2m)]$	$\frac{1}{6}\omega m^3(1+\gamma\delta+\gamma+\delta)$	
$\rho_{\Delta\bar{x}\Delta\bar{y}}(\pm 1)$			0.25

(*) $\Delta X = \alpha + \varepsilon_t + \gamma\varepsilon_{t-1}$

$\Delta Y = \beta + \eta_t + \delta\eta_{t-1}$

$E(\varepsilon_t\eta_t) = \omega$, $E(\varepsilon_t\eta_s) = 0$ for $t \neq s$.

TABLE 5	EFFECTS ON AUTO-COVARIANCES OF IMA(1,2) PROCESSES		
	any m	$m>>1$	$\lim_{m\to\infty}$
	(a) Under Sampling		
$R_{\Delta\bar{x}}(0)$	$R_{\Delta x}(0)[m+2(m-1)\rho_{\Delta x}(1)+2(m-2)\rho_{\Delta x}(2)]$	$mR_{\Delta x}(0)[1+2\rho_{\Delta x}(1)+2\rho_{\Delta x}(2)]$	
$R_{\Delta\bar{x}}(1)$	$R_{\Delta x}(0)[\rho_{\Delta x}(1)+2\rho_{\Delta x}(2)]$	$R_{\Delta x}(0)[\rho_{\Delta x}(1)+2\rho_{\Delta x}(2)]$	
$R_{\Delta\bar{x}}(2)$	0	0	
$\rho_{\Delta\bar{x}}(1)$			0
	(b) Under Temporal Aggregation		
$R_{\Delta\bar{x}}(0)$	$\frac{1}{3}R_{\Delta x}(0)[2m^3+m+4m(m^2-1)\rho_{\Delta x}(1)+2(2m^3+m-3)\rho_{\Delta x}(2)]$	$\frac{2}{3}m^3R_{\Delta x}(0)[1+2\rho_{\Delta x}(1)+2\rho_{\Delta x}(2)]$	
$R_{\Delta\bar{x}}(1)$	$\frac{1}{6}R_{\Delta x}(0)[m^3-m+2m(m^2+2)\rho_{\Delta x}(1)+2(m^3+11m-12)\rho_{\Delta x}(2)]$	$\frac{1}{6}m^3R_{\Delta x}(0)[1+2\rho_{\Delta x}(1)+2\rho_{\Delta x}(2)]$	
$R_{\Delta\bar{x}}(2)$	$R_{\Delta x}(2)$	$R_{\Delta x}(2)$	
$\rho_{\Delta\bar{x}}(1)$			0.25
$\rho_{\Delta\bar{x}}(2)$			0