A HOMOGENEOUS CLASS OF LINEAR ESTIMATORS AND STRONGER AITKEN ESTIMATOR

by

Eric Iksoon Im & Marcellus Snow

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ABSTRACT

We define a new class of linear estimators which includes as a subset all linear unbiased estimators. Subsequently, we establish Aitken estimator, the best linear unbiased estimator, further as the best in this larger class of linear estimators.

1. Introduction

Consider the linear regression model

$$y = X\beta + u \tag{1}$$

where y is a $(nx1)$ vector for dependent variable, X a (nxk) full column rank matrix for explanatory variables, β a $kx1$ parameter vector, and u a $(nx1)$ disturbance vector with zero mean and covariance matrix Ω of $rank(\Omega) \geq k$ (with the usual restrictive assumption, $E(X'u) = 0_{kx1}$, relaxed).

Writing linear estimators of β in general as

$$\begin{aligned} b &= Ay \\ &= AX\beta + Au \quad \text{for all } A \in \mathbb{R}^{kxn}, \end{aligned} \tag{2}$$

the mean squared error matrix of b [Magnus and Neudecker, p.285] becomes

$$MSE(b) = [AX-I]\beta\beta'[AX-I]' + A\Omega A'. \tag{3}$$

Then, the theorems in Section 2 are in order for a 'homogeneous' class of linear estimators formally defined as follows:

Definition. The homogeneous class of linear estimators is defined by $b = Ay$ for which $AX = I_k$.

2. Theorems.

Theorem 1. The homogeneous class of linear estimators can be expressed in the form homogeneous, hence afore refered to as such, with Aitken estimator of β:

$$b_H = b_H(\Sigma) = (X'\Sigma X)^{-1}X'\Sigma y \tag{4}$$

for all $\Sigma \in \mathbb{R}^{n\times n}$ of rank k.

proof: The general solution of $AX = I_k$ for A can be written [Magnus and Neudecker, p.37] as

$$A = X^+ + Q - QXX^+ \qquad \text{for all } Q \in \mathbb{R}^{k\times n} . \tag{5}$$

Noting that $X = X(X'V^{-1}X)^+(X'V^{-1}X)$ in general where V is any positive definite matrix [Magnus and Neudecker, p.260], that $X^+X = I_k$ since X is of full column rank, and that $X'V^{-1}X$ is nonsingular, we can write $I_k = (X'V^{-1}X)^+X'V^{-1}X = (X'V^{-1}X)^{-1}X'V^{-1}X$. Hence, matrix A in (5) can be written as

$$\begin{aligned} A &= I_k A \\ &= (X'V^{-1}X)^{-1}X'V^{-1}X[X^+ + Q - QXX^+] \\ &= (X'V^{-1}X)^{-1}X'[V^{-1}XX^+ + V^{-1}XQ - V^{-1}XQXX^+] \\ &= (X'V^{-1}X)^{-1}X'\Sigma \qquad (6) \end{aligned}$$

where

$$\Sigma \equiv V^{-1}XX^+ + V^{-1}XQ - V^{-1}XQXX^+. \qquad (7)$$

Postmultiplying X to (7),

$$\Sigma X = V^{-1}XX^+X + V^{-1}XQX - V^{-1}XQXX^+X = V^{-1}X, \qquad (8)$$

hence $rank(\Sigma X) = rank(V^{-1}X) = k$ which implies $rank(\Sigma) \geq k$.

Rewriting (7) for Q (see Appendix A for details),

$$Q = (X'\Sigma X)^{-1}X'\Sigma - X^{+} - H(I_n - MM^{+}) \text{ for any } H \in \mathbb{R}^{kxn} \tag{9}$$

where $M = I - XX^{+}$.

Since Q can be any matrix in $\mathbb{R}^{kxn}$ as defined in (5), (9) must hold for any $\Sigma \in \mathbb{R}^{nxn}$ of *rank k*, which establishes the domain for Σ.

Substituting $\Sigma X = V^{-1}X$ from (8) into A in (6),

$$A = (X'\Sigma X)^{-1}X'\Sigma. \tag{10}$$

Substituting (10) into (2),

$$b_H(\Sigma) = (X'\Sigma X)^{-1}X'\Sigma y.$$

Theorem 2. In the context of (1), the Aitken estimator $b_H(\Omega^{+}) = (X'\Omega^{+}X)^{-1}X'\Omega^{+}y$ is the best homogeneous linear estimator in the sense that the mean squared error matrix of any other homogeneous linear estimator $b_H(\Sigma)$ exceeds that of Aitken estimator $b_H(\Omega^{+})$ by a positive semidefinite matrix: $MSE(b_H(\Sigma)) \geq MSE(b_H(\Omega^{+}))$ for all $b_H(\Sigma)$.

proof.

$$\Delta \equiv MSE(b_H(\Sigma)) - MSE(b_H(\Omega^+))$$

$$= (X'\Sigma X)^{-1}X'\Sigma\Omega\Sigma'X(X'\Sigma'X)^{-1} - (X'\Omega^+X)^{-1}$$

$$= (X'\Sigma X)^{-1}X'\Sigma[\Omega - X(X'\Omega^+X)^{-1}X']\Sigma'X(X'\Sigma X)^{-1}$$

$$\equiv (X'\Sigma X)^{-1}X'\Sigma W\Sigma'X(X'\Sigma'X)^{-1} \geq 0_{kxk} \quad \text{for all } \Sigma \tag{11}$$

noting $W \geq 0_{nxn}$ as shown in Appendix B.

Theorem 3. All linear unbiased estimators of β (B_{LUE}) is a subset of the homogeneous class of linear estimators of β (B_H): $B_{LUE} \subset B_H$.

proof. A necessary condition for b in eqn.(2) to be unbiased is $AX = I_k$, the condition which defines the homogeneous class of linear estimators. Hence, all linear unbiased estimators of β must be a subset of the homogeneous class of linear estimators.

Corollary (Generalized Aitken Theorem). Aitken estimator of β is the best linear unbiased estimator. (See Theil [pp.278-279] and Aitken.)

proof. By virtue of Theorem 2 and 3

$$MSE(b_H(\Omega^+)) \leq MSE(b_{LUE}) \quad \text{for all } b_{LUE} \in B_{LUE}. \tag{12}$$

Since $MSE(b_{LUE}) = COV(b_{LUE})$ for all $b_{LUE} \in B_{LUE}$, and in the context of Aitken Theorem $b_H(\Omega^+) \in B_{LUE}$, (12) reduces to

$$COV(b_H(\Omega^+)) \leq COV(b_{LUE}) \quad \text{for all } b_{LUE} \in B_{LUE} \tag{13}$$

which is the essence of the Generalized Aitken Theorem.

3. On Instrumental Variable Estimators.

Suppose that $E(X'u) \neq 0_{kx1}$ in (1). Then, instrumental variable estimators of β, which are unbiased, are homogeneous linear estimators of β for which $\Sigma = X(X'X)^{-1}Z$, Z denoting any instrumetal variable matrix of appropriate order. (k-class estimators, instrumental variable estimators, are also of the homogeneous class.) Since the instrumental variable estimators are of the homogeneous class, they are, though unbiased, inferior to biased Aitken estimator by MSE criterion as established in theorem 2.

4. Conclusion.

We have identified a larger class of linear estimators which includes as a subset the class of linear unbiased estimators, and have introduced a theorem which formally establishes the Aitken estimator as a stronger linear estimator than it has been known in econometrics; Aitken estimator is not only the best within the class of the linear unbiased estimators, but also the best within the homogeneous class which includes all linear unbiased estimators as a subset. In view of Theorem 2, Aitken estimator may be preferred in the context of $E(X'u) \neq 0_{kx1}$ to an unbiased linear estimator when the disturbance covariance matrix is estimable.

Appendix A

Premultiplying $(X'V^{-1}X)^{-1}X'$ to Σ in (7), and rearranging,

$$(X'V^{-1}X)^{-1}X\Sigma - X^{+} = Q(I - XX^{+}) \quad . \tag{A.1}$$

Postmultiplying idempotent matrix $M \equiv (I_n - XX^{+})$ to (A.1),

$$[(X'V^{-1}X)^{-1}X'\Sigma - X^{+}]M = QM^2 = QM. \tag{A.2}$$

Rewriting (A.2),

$$[(X'V^{-1}X)^{-1}X'\Sigma - X^{+} - Q]M = 0_{kxn}. \tag{A.3}$$

The general solution of (A.3) [Magnus and Neudecker, p.38] for $[(X'V^{-1}X)^{-1}X'\Sigma - X^{+} - Q]$,

$$[(X'V^{-1}X)^{-1}X'\Sigma - X^{+} - Q] = H(I_k - MM^{+}) \quad \text{for all } H \in \mathbb{R}^{kxn}. \tag{A.4}$$

Writing (A.4) for Q, and substituting $V^{-1}X = \Sigma X$ from (7),

$$Q = (X'\Sigma X)^{-1}X'\Sigma - X^{+} - H(I_n - MM^{+}). \tag{A.5}$$

Appendix B

There exists matrix $S(nxr)$ for Ω, $rank(\Omega) = r$, such that $S\Lambda = \Lambda S$ and $S'S = I_r$ where Λ is a diagonal matrix of order r consisting of non-zero eigenvalues of Ω. In addition, since $rank(\Omega) \geq rank(X) = k$, $\Omega\Omega^{+}X = X$ or equivalently $SS'X = X$.

Hence,

$$W = \Omega - X(X'\Omega^{+}X)^{-1}X'$$

$$= S\Lambda S' - SS'X(X'S\Lambda^{-1}S'X)^{-1}X'SS'$$

$$= S[\Lambda - \tilde{X}(\tilde{X}'\Lambda^{-1}\tilde{X})^{-1}\tilde{X}']S' \qquad (\tilde{X} \equiv S'X)$$

$$= S\Lambda^{1/2}[I_r - \Lambda^{-1/2}\tilde{X}(\tilde{X}'\Lambda^{-1}\tilde{X})^{-1}\tilde{X}'\Lambda^{-1/2}]\Lambda^{1/2}S'$$

$$\equiv S\Lambda^{1/2}\tilde{W}\Lambda^{1/2}S' \geq 0_{nxn} \qquad (\because\ \tilde{W} = \tilde{W}^2 \geq 0_{rxr}) \ .$$

References

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