A Stronger Schur's Theorem and Its Application in Econometric Theory: A Note

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1. INTRODUCTION.

Schur (1911) established global bounds for eigenvalues for Hadamard product, which was discussed later by Styan (1973) and Horn (1990) in their respective survey articles on Hadamard product. However, the global bounds established by Schur are not narrow enough to be useful under certain circumstances. In this note, we establish the narrower bounds, and discuss an econometric example to demonstrate its usefulness in establishing consistency of an estimator.

2. DERIVATION OF UPPER AND LOWER BOUNDS FOR EIGENVALUES.

Writing a Hadamard product,

$$H = H_1 \odot H_2 \tag{1}$$

where $H_1(TxT)$, $H_2(TxT)$ $>$ 0 (i.e., positive-definite).

Then, we can rewrite H as

$$H = S'(H_1 \otimes H_2)S \tag{2}$$

where $\otimes$ denotes the Kronecker product as usual, and S is the T^2xT

matrix the *i-th* column of which has *1* in its $((i-1)T + i)th$ position and *0* elsewhere (See Amemiya, p 462).

Define matrices W_1 and W_2 :

$$W_1 = C_1 \otimes I_T; \quad W_2 = I_T \otimes C_2 \tag{3}$$

where C_1 and C_2 are defined such that

$$C_1' H_1 C_1 = \Lambda_1; \quad C_1' C_1 = C_1 C_1' = I_T; \quad C_2' H_2 C_2 = \Lambda_2; \quad C_2' C_2 = C_2 C_2' = I_T$$

in which Λ_1 and Λ_2 are *TxT* diagonal matrices consisting of eigenvalues of H_1 and H_2, respectively.

Hence, we can rewrite H in (2) as

$$H = S'(H_1 \otimes H_2)S = S'W_1W_1'(H_1 \otimes H_2)W_1W_1'S = S'W_1(\Lambda_1 \otimes H_2)W_1'S \ . \tag{4}$$

Noting that $\Lambda_1 \otimes H_2$ is positive-definite (without the proof),

$$\lambda_{\min}(H_1) \cdot (I_T \otimes H_2) \le \Lambda_1 \otimes H_2 \le \lambda_{\max}(H_1) \cdot (I_T \otimes H_2)$$

where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$, respectively, are minimum and maximum eigenvalues of the argument matrix, hence

$$\lambda_{\min}(H_1)\cdot S'W_1\left(I_T\otimes H_2\right)W_1'S \;\le\; H \;\le\; \lambda_{\max}(H_1)\cdot S'W\left(I_T\otimes H_2\right)W'S \;. \tag{5}$$

Since

$$S'W_1\left(I_T\otimes H_2\right)W_1'S = S'\left(I_T\otimes H_2\right)S = I_T\odot H_2 = \mathrm{A}(H_2)$$

where $\mathrm{A}(\cdot)$ is the diagonal matrix consisting of diagonal elements of the argument matrix,

we can rewrite (5) as

$$\lambda_{\min}(H_1)\cdot\mathrm{A}(H_2) \;\le\; H \;\le\; \lambda_{\max}(H_1)\cdot\mathrm{A}(H_2) \;. \tag{6}$$

Based on Amemiya (p. 460), (6) implies that

$$\lambda_{\min}(H_1)\cdot\alpha_i(H_2) \;\le\; \lambda_i(H) \;\le\; \lambda_{\max}(H_1)\cdot\alpha_i(H_2) \tag{7}$$

where $\alpha_i(\cdot)$ denotes the *i-th* largest diagonal element of the argument matrix.

Subscripts *1* and *2* in (7) are interchangeable, which would be the case if we replaced W_1 in (4) with W_2. Therefore, generalizing (7),

$$\lambda_{\min}(H_r)\cdot\alpha_i(H_s) \le \lambda_i(H) \le \lambda_{\max}(H_r)\cdot\alpha_i(H_s) \qquad (r,s = 1,2; \quad r \neq s) \quad (8)$$

Summmarizing what has been discussed so far formally as a theorem,

(Stronger Schur's Theorem): Let $H = H_1 \odot H_2$ where both H_1 and H_2 are TxT positive definite matrices. Then,

$$\lambda_{\min}(H_r)\cdot\alpha_i(H_s) \le \lambda_i(H) \le \lambda_{\max}(H_r)\cdot\alpha_i(H_s) \qquad (r,s = 1,2; \quad r \neq s)$$

where $\lambda_{\min}(\cdot)$ denotes the eigenvalues of the argument, and $\alpha_i(\cdot)$ the *i-th* largest diagonal element of the argument.

From the above theorem readily follows the Schur's Theorem (1911) as a corollary:

$$\lambda_{\min}(H_r)\cdot\alpha_{\min}(H_s) \le \lambda_i(H) \le \lambda_{\max}(H_r)\cdot\alpha_{\max}(H_s) \qquad (r,s = 1,2; \quad r \neq s) \quad (9)$$

discussed in details in Styan (1973).

The lower and upper bounds in (8) defines narrower bounds for eigenvalues than established by Schur, therefore more useful.

3. AN APPLICATION.

The stochastic parameter variation model discussed in Cooley and Prescott (1978), for example, has the disturbance covariance matrix which includes a Hadamard product of two (non-identity) positive-definite matrices. The model, after a linear transformation, can be expressed as

$$y = X\beta_{T+1} + \mu \tag{10}$$

for which

$$E(\mu\mu') = \sigma^2\Omega \;;$$
$$\Omega = \gamma Q + (1-\gamma)I \tag{11}$$

where

$$Q = A \odot R;\;\; A = [a_{ij}];\; a_{ij} = \min(T-i+1,\, T-j+1) \quad (i,j = 1,\dots,T)\;;$$
$$A = [a_{ij}];\; a_{ij} = \min(T-i+1,\, T-j+1)\;;$$
$$R = [r_{ij}];\;\; r_{ij} = \frac{x_i'\Sigma_v x_j}{\sqrt{x_i'\Sigma_u x_i}\sqrt{x_j'\Sigma_u x_j}};\;\; \Sigma_v = \Sigma_u = \Sigma\;.$$

Suppose we are interested in finiding the order of $S = \sum_{i=1}^{T} \lambda_i(\Omega^{-1})$.

By virtue of (8),

$$\lambda_i(Q) \;\le\; \lambda_{\max}(R)\cdot\alpha_i(A) = \lambda_{\max}(R)\cdot i\;, \tag{12}$$

and eigenvalues of Ω in (11) can be written as

$$\lambda_i(\Omega) = \gamma\lambda_i(Q) + (1-\gamma) \quad . \tag{13}$$

Based on (12) and (13), we can define the lower bounds for $\lambda_i(\Omega)$ as

$$\frac{1}{\lambda_i(\Omega)} \geq \frac{1}{\gamma\lambda_{\max}(R)\,i + (1-\gamma)} \quad . \tag{14}$$

Noting that

$$S = \Sigma\lambda(\Omega^{-1}) = \Sigma \frac{1}{\lambda_i(\Omega)} \; ;$$
$$\lambda_{\max}(R) \leq \Sigma\lambda_i(R) = tr(R) = T \; ;$$
$$0 \leq \gamma \leq 1 \; ,$$

from (14) readily follows that

$$S \geq \sum_{1}^{T} \frac{1}{\gamma\lambda_{\max}(R)\,i + (1-\gamma)} \equiv S_L \tag{15}$$

A simple analytical expression for S_L is impossible for small sample T, hence we have to consider S_L for asymptotic T. Denoting the asymptotic T by τ ,

$$\lim_{T\to\infty} S_L = \lim_{T\to\infty} \Sigma \frac{1}{\gamma\lambda_{\max}(R)\cdot i + (1-\gamma)}$$

$$= \int_{1/\tau}^{1} \frac{1}{\gamma\lambda_{max}x + \frac{1-\gamma}{\tau}}\, dx$$

$$= \left[\frac{1}{\gamma\lambda_{max}(R)} \ln\left(\gamma\lambda_{max}(R)\,x + \frac{1-\gamma}{\tau}\right)\right]_{1/\tau}^{1}$$

$$= \frac{1}{\gamma\lambda_{max}(R)} \ln\left[\frac{\gamma\lambda_{max}(R) + \frac{1-\gamma}{\tau}}{\gamma\lambda_{max}(R)\,\frac{1}{\tau} + \frac{1-\gamma}{\tau}}\right]$$

$$= \frac{1}{\gamma\lambda_{max}(R)} \ln\left[\tau \cdot \frac{\gamma\lambda_{max}(R) + \frac{1-\gamma}{\tau}}{\gamma\lambda_{max}(R) + 1-\gamma}\right] \equiv S_L^{*} \ . \tag{16}$$

Matrix R in (16) is basically a lag correlation coefficient matrix in which diagonal elements are unity and off-diagonals less than unity but taper off as the displacement increases, hence $\lambda_{max}(R)$ is likely to be a finite number. Therefore, assuming that $\lambda_{max}(R)$ is finite, we can consider the order of S_L^{*} over the range of γ :

(a). If $\gamma = 1$, then from (15) readily follows

$$S_L^{*} = \frac{1}{\lambda_{max}(R)} \ln\tau = O(\ln\tau) = \infty \ ; \tag{17}$$

(b). If $0 < \gamma < 1$, then also from (16) readily follows

$$S_L^{*} = \frac{1}{\gamma\lambda_{max}} \ln\left(\tau\left[\frac{\gamma\lambda_{max}(R)}{\gamma\lambda_{max}(R)+1}\right]\right) = O(\ln\tau) = \infty \ ; \tag{18}$$

(c). If $\gamma = 0$, (16) is not defined. As an alternative, we can consider S_L^* as $\gamma \to 0^+$. Defining

$$\delta = \frac{\gamma}{1-\gamma}\lambda_{max}(R)\tau \quad , \tag{19}$$

we can rewrite (16) as

$$S_L^* = \tau \ln\left[\frac{(1-\gamma)(1+\delta)}{\gamma\lambda_{max}(R) + (1-\gamma)}\right]^{1/\delta} \quad , \tag{20}$$

from which follows

$$\begin{aligned} \lim_{\gamma\to 0} S_L^* &= \tau \lim_{\gamma\to 0} \ln(1 + \delta)^{1/\delta} \\ &= \tau \lim_{\delta\to 0} \ln(1 + \delta)^{-1/\delta} \\ &= \tau \ln e \\ &= \tau = \infty \ . \end{aligned} \tag{21}$$

Based on the order of S_L^* in (16), we may state in general that S_L is in the order of *lnT* to *T*, which completes the proof that $\hat{\beta}_{gls}$ is consistent.

3. CONCLUDING REMARK.

The global upper bound for eigenvalues as established in Schur's Theorem in (9) does not allow for the lower bound narrow enough to establish the consistency of $\bar{\hat{\beta}}_{gls}$, whereas the narrower bounds introduced in this note does.

However, if both $\lambda_{max}(R)$ and $\lambda_{min}(R)$ are are finite or both in the same order, then the order of S is in the same order as the upper and lower bounds for S. Otherwise, the order of the lower bound for S may not be exactly in the same order as S. If that is the case, the narrower bounds introduced may not be as useful.

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