Firm's R & D Behavior Under Rational Expectations *

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Abstract

This paper formulates dynamic R&D investment decisions of private firms as an optimal stochastic control problem. It derives explicitly R&D investment decision rule and the cross equations parameter restrictions imposed by the rational expectations hypothesis, using the Riccati equations only and not requiring the use of Wiener-Kolmogorov prediction formula. Identification and estimation of the structural parameters are essential for evaluating policies such as R&D subsidies, firm size, market concentration so that the evaluations of these policies stand against Lucas critique. We find conditions under which the structural parameters are identified; we then discuss econometric procedures for using aggregate time series data or panel data on firms to deal with unobserved technological knowledge, to estimate the structural parameters, and to test the model.

Keywords: *Research and development, rational expectations, stochastic control*

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1 Introduction

It has now been well established that technological change is a major source of growth. As in the case of physical capital, returns on industrial research and development (R&D) investments in the private sector will depend upon the evolution of market conditions and public policies among other factors. Thus the Lucas critique [1976] on policy evaluation applies to R&D decisions, namely a firm's R&D investment decisions under uncertainty will depend upon its expectations about future market conditions and policy changes, and therefore its R&D decision rule will react to the changes in the stochastic processes of these factors. The point of the critique is that instead of estimating a R&D decision rule by throwing in arbitrarily some policy variables as regressors, one should model and estimate the parameters of the firm's objective function and the stochastic processes that constitute the environment in which the firm operates. To that end, we need tractable dynamic economic models of R&D investment decisions which lend to estimation and testing using available econometric techniques. Hansen and Sargent [1981] among others began such a line of research to model aggregate labor supply decisions over time of a representative agent. We follow their lead to provide a tractable model of R&D along the same line.

What makes tractable modeling of R&D investments difficult is that the output of this activity, technological knowledge, possesses many properties of a commodity for which standard economic theory fails. For instance, knowledge is an intangible, indivisible and inappropriable (i.e., difficult to institute a property right) commodity and exhibits externalities in its production and use. Therefore, unlike the physical capital, the market prices for knowledge do not exist which could guide R&D investment decisions (Arrow [1962], Griliches[1979], Nelson [1982]). The rate of accumulation of technological knowledge of a firm acquired by means of R&D investments will depend upon the following factors:

(1) Firm size, intensity of rivalry or competition (Schumpeter [1934, 1950]).

(2) Complete uncertainty about the profitability of a new product, if it is a product innovation, and partial uncertainty about the shifts in demand for the product if it is a process innovation (Schmookler [1966]).

(3) R&D capabilities or "strength of knowledge" for efficient R&D search for firms having different "science bases" (Rosenberg [1976], Nelson [1982], Nelson and Winter [1977,1978], Evenson and Kislev [1976]).

(4) Government policies such as R&D tax credits, MRTP (monopoly restrictive trade practices), licensing schemes, patent laws, investments in basic research affecting (1)-(3).

Theoretical models of R&D have considered most of the above aspects of technological knowledge, and studied the effects of government policies on R&D subsidies, market concentration, firm size (Dasgupta and Stiglitz [1980a&b among others, see Kamien and Schwartz [1981] for a survey of these papers). The empirical research, on the other hand, has been carried out mainly in two lines ignoring many of the above aspects of R&D. One set of studies is concerned with testing the Schumpeterian hypothesis regarding the effects of firm size and intensity of rivalry on the pace of R&D investments within a static framework (see Levin and Reiss [1984], Kamien and Schwartz [1981] for an account of these studies). The other set of studies is concerned with the effects of R&D expenditures on productivity growth (Griliches [1984], Mohnen [1992], Mairesse and Sassenou [1991], Raut [1995] for accounts of these studies for developed and developing countries). Although many studies are directed toward policy analysis, these studies do not formulate R&D investments using a dynamic economic model and then estimate the model parameters.[1]

In this paper we present a dynamic economic model of R&D investments that incorporates the above aspects of technological knowledge. We explicitly model the process of knowledge creation, as in Griliches [1979], and Pakes and Griliches [1984]. To impute a value to technological knowledge in each period, we assume that the timing of an innovation is unknown; however, the higher is the stock of knowledge, the higher is the probability of its taking place in any period. By applying techniques from statistical decision theory to this setup, we impute a value to stock of technological knowledge. We then show that the firm's R&D decision problem could be represented by an optimal stochastic control problem (see section 2, for details).

Hansen and Sargent [1981] gave up dynamic programming method for solving their cost-of-adjustment model of labor supply decisions on the grounds that the matrix Riccati equation did not lead to a closed form solution. They proposed an alternative method that uses Euler equation, Transversality condition and Wiener-Kolmogorov prediction formula to compute a close form optimal decision. I show

[1]Pakes [1984], however, goes a step closer in this direction; instead of deriving the reduced form solution with cross equations restrictions imposed by rational expectations, he, however, parameterizes the reduced form solution for estimation.

in this paper, however, that when the control variable is one dimensional (which was the case in their model too) it is possible to derive a close form solution solely from the matrix Riccati equation, and thus the Wiener- Kolmogorov prediction formula is not required for this purpose; this is done in section 3.

In section 4, we investigate the identification of the parameters of the objective function and the stochastic processes of the environment. We show that while the structural parameters are unidentified when the environment is represented by a first order auto-regressive process, when the environment is represented by an autoregressive process of order two or higher, the system is generally over identified; and thus one can estimate all the structural parameters. In section 5, we describe how the decision rule and the cross equation restrictions change when we include other exogenous variables in the information set that Granger cause the stochastic processes representing the environment.

Previous sections are based on the assumption that technological knowledge is observable. In section 6, we relax this assumption and give a closed form solution based on noisy measurements of technological knowledge. In section 7, we consider various econometric strategies that could be adopted to estimate the structural model and use the overidentified restrictions to test the model. I have estimated in Raut [1988] an unrestricted reduced form decision rule using panel data of Indian private firms. Using data from developed countries, further empirical research along this line will shed more light about the actual decision making process of R&D investments of the private firms.

2 The Basic Model

2.1 Technological Knowledge

There are at least three different ways in which technological knowledge has been conceptualized in the literature. Arrow[1962] defines technological knowledge as information about the states of nature. In his framework, investment in R & D is visualized as acquiring more knowledge about the states of nature to improve one's subjective beliefs about the possibility of reaping an innovation, based on Baysian learning mechanism.

Nelson [1982] treats technological knowledge as "capability for efficient search". In his models, R&D is viewed as search for a given target, say for instance, a product innovation or a process innovation. The search could be targeted in different directions but with stochastic outcomes. R&D investments are related to the number of elements that are expected to be drawn for efficient searching before the desired outcome is achieved. The main point of this notion of technological knowledge is that it is the strength of knowledge that determines how much R&D efforts are expected to be successful as opposed to Schmookler's viewpoint [1966] that the pay-off determines R&D investments. Nelson [1982], and Nelson and Winter[1982] formalized and gave more operational content to this line of reasoning that originated in the works of Rosenberg [1976]. Nelson also studied the relationship between knowledge and innovation explicating the public good aspect, the externalities in the production, and to the sources of technological knowledge.

Griliches[1979] gave more empirical content to his definition of technological knowledge. Griliches [1979,1984] treated the stock of technological knowledge as one of the factors of production analogous to stock of physical capital. Like capital stock, it depreciates and becomes obsolete over time, but can replenish over time with R&D investments. He used a production function framework to study the contribution of R&D on productivity growth at the firm level for U.S. firms. To model the spill-over effect empirically, he introduced the notion of technological distance between two firms.

My definition of technological knowledge draws from all three notions. I view accumulation of technological knowledge as acquisition of more information about the states of nature–information about product improvements or process improvements; I also view it as a deliberate economic activity similar to investment in physical capital. A set of R&D inputs adds to the stock of knowledge which might be immediately used or might be useful for further information production. However, unlike in the case of investment in physical capital, I assume here that the marginal rate, b, at which a unit of R&D adds to the stock of knowledge varies from industry to industry depending on the R&D capability or strength of knowledge or the science base of that industry. There are various sources for spill-over effects, e.g., government's investment in basic research, (which brings technological change to the process of technological change, so to speak), technological knowledge of other domestic or foreign firms, the strength of which depends on the patent law. I assume all these constitute a constant amount of spill over knowledge in each period. More formally,

$$z_{1t+1} = a_1 z_{1t} + bR_t + c_1 + w_{1t},\ t \geq 0 \tag{1}$$

where,

z_{1t}	=	our firm's stock of knowledge at the beginning of period t
R_t	=	R & D investment of the firm in period t
$1 - a_1$	=	depreciation rate for knowledge
b	=	technological capability or a measure of strength of knowledge
c_1	=	a constant measuring the spill-over effect
w_{1t}	=	random shock in period t.

The specification (1) of the technology of technological knowledge production is general enough to incorporate various empirical findings on differential lagged effects of R&D investments on stock of knowledge.

2.2 Valuation of technological Knowledge

Technological knowledge is an intangible, indivisible, inappropriable, i.e., difficult to institute a property right, and involves externalities in production and its use. Patent law is a legal protection assuring only a partial appropriation. There do not exist markets for technological knowledge (Arrow [1962]). Following the strategy of valuation of information in statistical decision theory, we impute an indirect private value to a stock of technological knowledge in the following way.

The timing of innovation our firm is pursuing is not known, but the likelihood of its taking place in any period is higher, the greater is the stock of accumulated knowledge in the beginning of that period. Let $P(z_{1t})$ be the probability that the firm will reap the innovation in period t if its stock of knowledge is z_{1t}, given that it has not achieved it yet. Various forms for $P(.)$ are plausible. One would like it to satisfy the following: $dP(z)/dz > 0$ and $d^2P(z)/dz^2 < 0$. The reasonable forms for P(.) are as follows:

$$P(z) = 1 - e^{-\gamma z},\ z \geq 0,\ \gamma > 0 \tag{2}$$

and

$$P(z) = \frac{1}{\mu - \gamma}.(\mu z - \gamma z^2),\ 0 < z \leq 1,\ \mu > \gamma > 0 \tag{3}$$

The value of an innovation at t will depend in a number of ways on the firm size, z_{2t}, intensity of rivalry, z_{3t}, and the market condition, or the profitability from the current line of research, ξ_t. Market concentration or intensity of rivalry is indeed an industry level attribute. More rivals in an industry may entail a higher chance for imitation of an innovated product or process and also a higher chance for another firm's innovation to arrive before the current innovation has reaped its maximum monopoly rent. Moreover, more rivals may reduce the market share of a firm. All these lead to lower value for an innovation.[2]

The effect of firm size on the value of technological knowledge may come through different channels. Following Nelson's [1959] interpretation, I argue that the larger firms, on the one hand, having already established name and reputation in the market, can appropriate the benefit of an innovation by easy market penetration, and on the other hand, having more product diversification, could use the accumulated knowledge in more than one line of business. Therefore, the larger firms may envisage a bigger return from a given stock of technological knowledge than the smaller ones.[3]

Another important factor in the determination of value of technological knowledge is completely unknown demand for new products in the case of product innovation and shift in the demand in the case of process innovation. Higher are these uncertainties, the lower will be the value of technological knowledge. This is sometimes referred as Schmookler's hypothesis or demand pull or market opportunity hypothesis.

Taking into account all the above factors in valuation of knowledge, let us denote the value of the innovation by

$$\eta(z_{2t}, z_{3t}, \xi_t) \tag{4}$$

For simplicity I am assuming here that (4) gives the present value at time t of the stream of cash-flows that the innovation will bring, and it depends only on the market condition prevailing at that time, but not on the future market conditions. For instance, this will be the case if the innovation is patented and sold to another firm for an amount of royalty payments, whose value is determined by the market conditions prevailing then.[4]

If R_t is the R&D input used in period t, and if we assume that the cost of R&D is quadratic in input use, one period expected reward from a stock of knowledge, z_{1t}, in period t is given by

$$v(z_{1t}) = \eta(z_{2t}, z_{3t}, \xi_t)P(z_{1t}) - 0.[1 - P(z_{1t})] - \theta R_t^2 \tag{5}$$

[2]Also greater monopoly power reduces the incentive for innovation as the firm with monopoly power can continue to earn the monopoly rent without venturing into a new technological innovation. It is generally argued that an intermediate level of market concentration is most conducive to rapid technological innovation.

[3]It should, however, be noted that the smaller firms are not necessarily restricted to use their knowledge only in their own production units as they can always sell it to another firm with licensing arrangements.

[4]Kamien and Schwartz [1981], explicitly modeled rivalry using a subjective hazard function, and then derived a functional relationship between rivalry and the present value of an innovation.

plus a stock of technological knowledge, z_{1t+1} as given by (1). If we further assume that after reaping the targeted innovation, our firm will venture into another innovation that will use the knowledge of the previous pursuit, then the firm faces an infinite horizon for its R&D investment decisions. In that case, given the sequences, $\{z_{2t}\}$, $\{z_{3t}\}$, and $\{\xi_t\}$, that characterize the environment facing our firm, the expected value of a sequence of technological knowledge $\{z_{1t}\}$ obtained by using $\{R_t\}$ is given by

$$V_0 = E_0 \sum_{t=0}^{\infty} \beta^t \left(\eta(z_{2t}, z_{3t}, \xi_t) P(z_{1t}) - \theta R_t^2 \right) \tag{6}$$

where $E_t(x)$ denotes the conditional expectation of x given information set Ω_t. For our purpose, we will have the following linear specification of the reward function.

$$\eta(z_{2t}, z_{3t}, \xi_t) = r_0 + r_1 \xi_t + r_2 z_{2t} + r_3 z_{3t} \tag{7}$$

where, $r_0, r_1, r_2 > 0$ and $r_3 < 0$. Substituting (3) in (7) and disregarding all terms with powers greater than two, one gets

$$\eta(z_{2t}, z_{3t}, \xi_t) \cdot P(z_{1t}) = (r_0 + r_1 \xi_t + r_2 z_{2t} + r_3 z_{3t}) \cdot \frac{\mu}{\mu - \gamma} \cdot z_{1t} - \frac{r_0 \gamma}{\mu - \gamma} \cdot z_{1t}^2$$

Regarding $r_0 + r_1 \xi = \zeta$, we have from (5)

$$v(z_{1t}) = Z_t' Q Z_t + H R_t^2 \tag{8}$$

where, $Q = (q_{ij})_{i,j=1,\dots 4}$ $H = -\theta, q_{11} = -r_0\gamma/(\mu-\gamma), q_{12} = r_2\mu(\mu-\gamma), q_{13} = r_3\mu/(\mu-\gamma)$ $q_{14} = \mu/(\mu-\gamma)$ other q_{ij}'s are zero, and $Z_t = (z_{1t}, z_{2t}, z_{3t}, \zeta_t)'$. I further assume that z_{2t}, z_{3t}, and ζ_t follow a first[5] order auto-regressive process, given by

$$\left.\begin{aligned} Z_{2t+1} &= a_2 Z_{2t} + c_2 + w_{2t} \\ Z_{3t+1} &= a_3 Z_{3t} + c_3 + w_{3t} \\ \zeta_{t+1} &= a_4 \zeta_{4t} + c_4 + w_{4t} \end{aligned}\right\} \tag{9}$$

where (w_{2t}, w_{3t}, w_{4t}) is a three dimensional vector of white noise process. Writing (1) (11) together, we have,

$$Z_{t+1} = A Z_t + B R_t + c + w_t \tag{10}$$

[5] In later section we will consider higher order auto-regressive processes.

where

$$A = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix}, \; B = \begin{pmatrix} b \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}, \; w_t = \begin{pmatrix} w_{1t} \\ w_{2t} \\ w_{3t} \\ w_{4t} \end{pmatrix}$$

We assume w_t to be a 4-dimensional white noise process.

I am assuming here for simplicity that R&D activities affect neither the firm size nor the intensity of rivalry. While this assumption is innocuous in the short-run, [6] but for a medium to long-run analysis this may not be the case, see for instance, Landes [1969] for historical evidence.

2.3 Firm's problem

We assume that manager of our firm knows the parameters of his objective function, (6), and the parameters of the stochastic processes (10). At the beginning of each period, t, he observes the realization of the variables in his information set, Ω_t. The variables in his information set include any stochastic process that Granger causes either z_{1t}, z_{2t}, z_{3t}, or ζ_t. I assume for now that the manager of our firm can observe z_{1t}. We will relax this assumption in a later section. Given Ω_t in period t, he chooses a R&D investment R_t so as to maximize (6) subject to (10).

The corresponding Bellman's equation for a slightly general case is as follows:

$$V_t(Z_t) = \max_{R_t} \Big(Z_t^{'} Q Z_t + q' Z_t + R_t^{'} H R_t + h' R_t + \beta E_t V_{t+1}(A Z_t + B R_t + c + w_t) \Big) \tag{11}$$

In our particular case, q and h are zero vectors. Solution of (11) gives the optimal R_t as a function of Ω_t. We find solution in the next section.

3 Closed form solution to firm's optimization problem

For now we assume that stock of knowledge is observable. We will relax this assumption later. A *stationary solution* is a function, $R_t = \mu(Z_t)$ for all $t \geq 0$. Following Bertsekas [1976], and Chow [1975,1981], we show that under certain conditions[7] on A, B, c, Q, and H, there exists an optimal stationary solution to (11) as stated in the following proposition:

[6] However, see Levin [1981], and Levin and Reiss [1984] for studies of the simultaneity of R&D expenditures and market concentration in a static framework.

[7] such as controllability and observability, see Bertesekas [1976] for details

Proposition 1 *The optimal stationary solution of (11) is given by*

$$R_t = -(GZ_t + g) \tag{12}$$

where

$$G = \beta(H + \beta B'KB)^{-1}B'KA \tag{13}$$

$$g = (H + \beta B'KB)^{-1}\left(\beta B'Kc + \frac{\beta}{2}B'k + \frac{h}{2}\right) \tag{14}$$

K is a positive definite solution of the following matrix Riccati equation:

$$K = Q + \beta A'[K - \beta KB(H + \beta B'KB)^{-1}B'K]A \tag{15}$$

and k is the solution to the following vector Riccati equation:

$$k'(I-\beta(A-BG)) = q'+2g'(H+\beta B'KB)G+2\beta c'K(A-BG)-h'G-2\beta g'B'KA \tag{16}$$

Proof: We guess a solution for V of the form:

$$V(Z) = Z'KZ + k'Z + \kappa \tag{17}$$

where K is a positive definite matrix, k is vector of positive numbers and κ is a non-negative real number.

Substituting (17) in (11), we have

$$\begin{aligned} V(Z_t) \quad = \quad & \max_{R_t} \{Z_t'QZ_t + q'Z_t + R_t'HR_t + h'R_t \\ & +\beta\,[Z_t'A'KAZ_t + 2Z_t'A'KBR_t + 2Z_t'A'Kc \\ & +R_t'B'KBR_t + 2R_t'B'Kc + c'Kc \\ & +E_t(w_t'Kw_t) \\ & +k'AZ_t + k'BR_t + k'c + \kappa\,]\} \end{aligned} \tag{18}$$

The first order condition yields equation (12). To find the solution for K, k, κ, we substitute the optimal value of R_t from (12) in (18) and the value of $V(Z)$ from (17) in the left hand side of (18) and then collecting terms for the quadratic terms $Z_t^{'}KZ_t$, the linear terms $k^{'}Z_t$ and the constant term, we get (15) and (16) respectively.

Q.E.D

Note that the vector Riccati equation (16) involves K, G and g, whereas the matrix Riccati equation K does not involve k and g. To find the close form optimal decision rule, (12), note that

$$\beta B'KB + H = b^2\beta k_{11} + \theta$$

and

$$B'KA = (bk_{11}a_1, ..., bk_{14}a_4)$$

Substituting these in (12), we have

$$R_t = -\frac{b\beta}{b^2\beta k_{11} + \theta}(k_{11}a_1z_{1t} + k_{12}a_2z_{2t} + ... + k_{14}a_4z_{4t}) - g \qquad (19)$$

We now compute k_{11}, $k_{12}, ...k_{14}$ from the Riccati equation, (15):

$$\begin{aligned} K &= A'\left[\beta K - \tfrac{\beta^2 KBB'K}{\beta b^2 k_{11} + \theta}\right] A + Q \\ &= A'DA + Q \\ &= \begin{pmatrix} a_1^2 d_{11} & a_1 d_{12} a_2 & ... & a_1 d_{14} a_4 \\ ... & ... & ... & ... \\ a_4 d_{42} a_1 & a_4 d_{42} a_2 & ... & a_4^2 d_4 \end{pmatrix} + Q \end{aligned} \qquad (20)$$

where,

$$\begin{aligned} D &= (d_{ij})_{i,j=1,2,3,4} \\ &= \beta K - \tfrac{\beta^2 KBB'K}{\beta b^2 k_{11} + \theta} \\ &= \beta K - K \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} K \end{aligned} \qquad (21)$$

where, $m = \frac{\beta^2 b^2}{\beta b^2 k_{11} + \theta}$. It is now easy to compute d_{ij}'s as follows:

$$d_{11} = \beta k_{11} - mk_{11}^2 = \frac{\theta\beta k_{11}}{b^2\beta k_{11} + \theta},\ \text{after substituting the value of } m$$

$$d_{12} = \beta k_{12} - m k_{11} k_{12} = \frac{\theta\beta k_{12}}{b^2\beta k_{11} + \theta}, \text{ after substituting the value of } m$$
$$\cdots\cdot$$

Substituting these in the right hand side of the last equality of (20) and then equating the matrix elements of both sides, we have

$$k_{11}\left(1 - \frac{a_1^2\theta\beta}{b^2\beta k_{11} + \theta}\right) = q_{11}, i.e., k_{11}(1 - a_1\lambda) = q_{11} \tag{22}$$

$$k_{12}\left(1 - \frac{a_1 a_2\theta\beta}{b^2\beta k_{11} + \theta}\right) = q_{12}, i.e., k_{12}(1 - a_2\lambda) = q_{12} \tag{23}$$
$$\cdots$$

where, $\lambda = \frac{a_1\theta\beta}{b^2\beta k_{11}+\theta}$. Substituting these in (19) and simplifying the expression (14) in a similar fashion, we have the following close form decision rule, (24) which is stated as a proposition.

Proposition 2 *A closed form solution to the firm's problem is given by*

$$R_t = -g + \alpha_1 Z_{1t} + \alpha_2 Z_{2t} + \alpha_3 Z_{3t} + \alpha_4 \zeta_t \tag{24}$$

where,

$$g = \frac{b\lambda}{a_1\theta(1-\lambda)} \sum_{j=1}^{4} c_j (1 - a_j\lambda)^{-1} q_{1j}$$

$$\alpha_i = -\frac{b\lambda}{a_1\theta} a_i (1 - a_i\lambda)^{-1} q_{1i}, \; i = 1, \ldots 4$$

$$\lambda = \frac{a_1\theta\beta}{b^2\beta k_{11} + \theta}$$

and k_{11} is a positive solution of the quadratic equation:

$$k_{11}\left(1 - \frac{\theta\beta a_1^2}{b^2\beta k_{11} + \theta}\right) = q_{11}$$

Equation (24) together with the system of equations for motion of the environment, (10), constitute the firm's decision rule. The assumption of rational expectations and a particular specification of the stochastic processes, (10) have generated cross equations parameter restrictions in the decision function of the firm. These restrictions are generally used for identification of the structural parameters and also for testing the rational expectations hypothesis assuming the model (10) is correct or for testing the specification of the model (10) assuming the rational expectations hypothesis is correct. We will take up the identification issues in the next section and estimation and testing issues in a later section.

4 Identification of parameters: need for more lags

The structural parameters are $\delta = (\theta, \beta, q_{1j},\ j = 1, 2, ...4)$: from the objective function; $\nu = (\sum, b, a_i', c_i, i = 1, 2, ...4)$: from the stochastic processes (10). We will assume for now that Z_{1t} is observable. Therefore, the second set of coefficients could be estimated from the system of equations (10). However, we will show that not all of the first set of parameters could be identified, and hence they could not be estimated from the observed data. To that end, note that from (24), we have

$$\sum_{j=1}^{4} c_j \alpha_j / a_j = (1 - \lambda) g \tag{25}$$

$$k_{11}(1 - a_1 \lambda) = q_{11} \tag{26}$$

$$\lambda = \frac{a_1 \theta \beta}{b^2 \beta k_{11} + \theta} \tag{27}$$

$$\alpha_1 = -\frac{\lambda b}{(1 - a_1 \lambda)\theta} q_{11} \tag{28}$$

Note that from (25) we can get an estimate of λ. Substituting the value of q_{11} from (28) in (26) and then substituting the value of k_{11} in (27) we get

$$\lambda = \frac{a_1 \beta}{1 + b^2 \beta \alpha_1 / \lambda} \tag{29}$$

From (29) we can get an estimate of β. But since θ cancelled out, we cannot identify it in this system. Therefore, the system (10) - (24) is under identified.

4.1 Higher Order Lags and Identification Problem

We would investigate here whether specification of a higher order auto-regressive process for (10) may ameliorate the under identification problem. We show that the parameters could indeed be identified. While for any general auto-regressive processes in (10), the closed form solution could easily be derived, for expositional ease, we assume here a third order auto-regressive process for (10), namely

$$\left.\begin{array}{rcl} Z_{1t+1} & = & a_1 Z_{1t} + bR_t + c_1 + w_{1t} \\ Z_{2t+1} & = & \delta_2(L) Z_{2t} + c_2 + w_{2t} \\ Z_{3t+1} & = & \delta_3(L) Z_{3t} + c_3 + w_{3t} \\ \zeta_{t+1} & = & \delta_4(L) \zeta_t + c_4 + w_{4t} \end{array}\right\} \tag{30}$$

where,

$$\delta_i(L) = a_{i1} + a_{i2}L + a_{i3}L^2, i = 2, 3, 4$$

and L is the lag operator, i.e., $LX_t = X_{t-1}$. Note that we still take the same form for Z_{1t}.

To derive the close form solution for the specification (30), we expand the state space Z_t to contain all the lag values of z's and then extend the definition of A, Q, B and c appropriately and proceed exactly the same way as we derived the solution in proposition 3.2. We sketch our derivation for the special case, where we assume that only z_{2t} follows a third order autoregressive process, and assume that there are only two other state variables, z_{1t} and z_{3t} both of which follow first order autoregressive process as in the previous model. The form of the close form solution when all state variables except z_{1t} follows third order autoregressive process will be transparent from the derivation of this special case. In our sketch, we basically note the differences with our earlier derivation. First of all, note that our new Z_t,A, B, and Q, C, and W_t are respectively

$$Z_t = \begin{pmatrix} z_{1t} \\ z_{2t} \\ z_{2t-1} \\ z_{2t-2} \\ z_{3t} \end{pmatrix} A = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & a_{21} & a_{22} & a_{23} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_3 \end{pmatrix}, B = \begin{pmatrix} b \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$Q = \begin{pmatrix} q_{11} & q_{12} & 0 & 0 & q_{13} \\ .. & .. & .. & .. & .. \\ & .. & & .. & \end{pmatrix}, c = \begin{pmatrix} c_1 \\ c_2 \\ 0 \\ 0 \\ c_3 \end{pmatrix}, w_t = \begin{pmatrix} w_{1t} \\ w_{2t} \\ 0 \\ 0 \\ w_{3t} \end{pmatrix}$$

Directly from the formula of G in (13), we have

$$G = \frac{-b\beta}{b^2\beta + \theta} [a_1 k_{11},\ a_{21}k_{12} + k_{13},\ a_{22}k_{12} + k_{14},\ a_{23}k_{12},\ a_3 k_{15}] \tag{31}$$

The above is parallel of equation (19). It is clear from the above that we need to compute only the first row of the Riccati matrix K. To that end, proceeding as in equations, (20) and ((21) of the previous model, we have the following five equations corresponding to the first row of the matrix equality in (15)

$$k_{11}(1 - a_1\lambda) = q_{11} \tag{32}$$

$$k_{12}(1 - \lambda a_{21}) - \lambda k_{13} = q_{12} \tag{33}$$

$$k_{13} - \lambda a_{22} k_{12} - \lambda k_{14} = 0 \tag{34}$$

$$k_{14} - \lambda a_{23} k_{12} = 0 \tag{35}$$

$$k_{15}(1 - a_3\lambda) = q_{13} \tag{36}$$

Note that as expected the equations (32) and (36) are as in the previous model. Equations (33)-(35) yield,

$$k_{12} = \frac{q_{12}}{1-\lambda\delta_2(\lambda)}, \; k_{13} = \frac{(\lambda a_{22}+\lambda^2 a_{23})q_{12}}{1-\lambda\delta_2(\lambda)}, \; k_{14} = \frac{\lambda a_{23} q_{12}}{1-\lambda\delta_2(\lambda)} \tag{37}$$

Let us denote the the reduced form parameters corresponding to the z_{2t} and its lag values by

$$\alpha_2(L) = \alpha_{21} + \alpha_{22}L + \alpha_{23}L^2$$

Substituting (37) in (31), we have

$$\alpha_{21} = -\frac{b\lambda q_{12}}{a_1\theta} \cdot \frac{\delta_2(\lambda)}{1 - \lambda\delta_2(\lambda)} \tag{38}$$

$$\alpha_{22} = -\frac{b\lambda q_{12}}{a_1\theta} \cdot \frac{a_{22} + \lambda a_{23}}{1 - \lambda\delta_2(\lambda)} \tag{39}$$

$$\alpha_{23} = -\frac{b\lambda q_{12}}{a_1\theta} \cdot \frac{a_{23}}{1 - \lambda\delta_2(\lambda)} \tag{40}$$

It is now clear how to derive these formulae for the general case. We state the general result in the following proposition:

Proposition 3 *A closed form solution of (11) and (30) is given by*

$$R_t = -g + \alpha_1 Z_{1t} + \alpha_2(L)Z_{2t} + \alpha_3(L)Z_{3t} + \alpha_4(L)\zeta_t \tag{41}$$

where

$$g = \frac{\lambda b}{a_1\theta}\left[1 + \frac{\alpha_1\lambda^2 b}{a_1^2\theta_1^2\beta(1-\beta(a_1+b\alpha_1)}\right]^{-1}\left(\frac{c_1 - \beta a_1 b}{1-\lambda a_1}q_{11} + \sum_{j=2}^{4}\frac{c_j q_{1j}}{1-\lambda\delta_j(\lambda)}\right) \tag{42}$$

$$\begin{aligned}\alpha_j(L) &= -\frac{b\lambda}{a_1\theta} \cdot \frac{a_{j1} + a_{j2}(\lambda + L) + a_{j3}(\lambda^2 + \lambda L + L^2)}{1 - a_{j1}\lambda - a_{j2}\lambda^2 - a_{j3}\lambda^3} \cdot q_{1j} \\ &= -\frac{b\lambda}{a_1\theta} q_{1j} \sum_{i=0}^{2} \frac{(\delta_j(\lambda)/\lambda^i))_+}{1 - \lambda\delta_j(\lambda)} L^i, \quad j = 2,3,4\end{aligned}$$

$()_+$ *denotes the* **annihilation operator** *that tells us to ignore negative powers of L; $\alpha_1, \lambda, k_{11}$ are as defined in (24) of proposition 3.2.*

Hansen and Sargent [1981] gave similar close form solution using Wiener - Kolmogorov prediction formula. However, in our case (41) and (42) have been derived directly from the matrix Riccati equation of the problem.

We now examine the identification of the structural parameters in (30)-(42).

Equations (38)-(40) imply

$$\frac{\alpha_{21}}{\alpha_{23}} - \lambda \frac{\alpha_{22}}{\alpha 23} = \frac{a_{21}}{a_{23}} \tag{43}$$

From equation (43), we can estimate λ. It is now clear that given λ, we can estimate β from (26)-(29). (Note that (26)-(29) are valid in this case also).

Also note that substituting in (42) the values of q_{11} from (28), q_{12} from (39) and q_{13} and q_{14} from the equations that parallel (39), we can get an estimate of θ. Now from (39) we get q_{12}, and from equations parallel to (39) for Z_{3t} and ζ_t we get q_{13} and q_{14}. Finally, from (28) we get q_{11}. So, all the structural parameters could be recovered in this case. Note that we have never used parallel of (38) and (40) from the other two variables, Z_{3t}, ζ_t in our identification strategy.

Therefore, the rational expectations hypothesis has imposed over identifying restrictions across equations.

5 Granger causality and choice of variables in Ω_t

So far implicitly we have been assuming that Ω_t contains only Z_{1t}, Z_{2t}, Z_{3t}, ζ_t and their lag values. In fact, we should include in Ω_t all observable variables that Granger cause either Z_{1t}, Z_{2t}, Z_{3t}, or ζ_t. In this section, we consider the nature of the close form solution and the cross equations restrictions that will be generated by the rational expectations hypothesis in such a case. For expositional ease, we continue to assume third order auto-regressive processes for Z_{2t}, Z_{3t} and ζ_t. Let us assume that we have an extra stochastic process X_{2t} which Granger causes Z_{2t} and which is related to Z_{2t} process as follows

$$\left.\begin{aligned} Z_{2t+1} &= a_2(L)Z_{2t} + \mu_2(L)X_{2t} \\ X_{2t+1} &= a_2'(L)Z_{2t} + \mu_2'(L)X_{2t}\zeta_{t+1} \end{aligned}\right\} \tag{44}$$

where

$$a_2(L) = a_{21} + a_{22}L + a_{23}L^2$$

and

$$\mu_2(L) = m_{21} + m_{22}L + m_{23}L^2$$

similarly $a'_2(L)$ and $\mu'_2(L)$ are defined.

Taking Z_{1t}, Z_{3t}, ζ_t as before, and expanding the state space variable Z_t to accommodate X_t and its lag, appropriately modifying the matrices, A, B, C and using the matrix Riccati equation we derive the close form solution as given in the following proposition:

$$R_t = -g + \alpha_1 Z_1 t + \alpha_2(L) \begin{pmatrix} Z_{2t} \\ X_{2t} \end{pmatrix} + \alpha_3(L)Z_{3t} + \alpha_4(L)\zeta_t \tag{45}$$

where g, α_1, $\alpha_3(L)$ and $\alpha_4(L)$ are as in (41), and $\alpha_2(L)$ is given by

$$\alpha_2(L) = \begin{pmatrix} \alpha_{21} + \alpha_{22}L + \alpha_{23}L^2 & \alpha'_{21} + \alpha'_{22}L + \alpha'_{23}L^2 \end{pmatrix}$$

$$\alpha_{21} = -\frac{\lambda b}{a_1\theta} \cdot \frac{a_2(\lambda) + a'_2(\lambda)\rho(\lambda)}{\psi(\lambda)} q_{12}$$

$$\alpha_{22} = -\frac{\lambda b}{a_1\theta} \cdot \frac{a_{22} + a'_{22}\rho(\lambda)}{\psi(\lambda)} q_{12}$$

$$\alpha_{23} = -\frac{\lambda b}{a_1\theta} \cdot \frac{a_{23} + a'_{23}\rho(\lambda)}{\psi(\lambda)} q_{12}$$

$$\rho(\lambda) = \frac{\lambda\mu(\lambda)}{1 - \lambda\mu'(\lambda)}$$

$$\psi(\lambda) = 1 - a(\lambda) - a'(\lambda)\rho(\lambda)$$

a'_{2j}'s are the same as α_{2j}'s with a_{2j}'s being replaced by m_{2j}'s.

The identification of structural parameters in this case follows the same steps as in the previous model. Notice that the above could be generalized for higher order auto-regressive processes and for other z-variables easily.

6 Unobserved Technological Knowledge

So far we have assumed that Z_{1t} is observable. We relax this assumption now. We assume instead that we have a set of "noisy - measurements", X_t's for Z_{1t}. Denote by $\hat{Z}_{1t} = E(Z_{1t}|X_t)$. It is well known in the control theory that the same closed form solutions hold if we replace Z_{1t} by $\hat{Z}_{1t}$ in (24) (41) and (45). However, the problem still remains how to evaluate $E(Z_{1t}|X_t)$.

Two approaches could be followed to estimate $E(Z_{1t}|X_t)$. One approach, used in the optimal control literature, is based on the Kalman-filtering formula (see Chow [1981] for an exposition of Kalman filtering). While this is an appropriate approach, it assumes that initial stock of knowledge is known, which is rather a strong assumption if we use panel data of firms. Even when we could obtain some noisy estimate of the initial stock of technological knowledge, Kalman filtering algorithm when combined with an algorithm of maximum likelihood estimation of the structural parameters become highly non-linear and may not converge in most cases.

An alternative approach followed by Griliches [1979] and Pakes and Griliches [1984] is to take changes in stock of knowledge at t as weighted sum of past five years' R & D investments and then relate it to the number of patents applied for by the firm in any period. Although, their purpose was not the estimation of stock of knowledge, their method could be adopted to generate an empirical measure of knowledge up to a scale factor. Actually, from their productivity analysis, one could get a direct estimate of the weights for different lags of R&D expenditures and hence a measure of knowledge with measurement errors (see Griliches [1979]). Following this line of research, we postulate that

$$Z_{1t} = \beta' X_t + u_t \tag{46}$$

where β is a vector of regression coefficients and X_t include past R & D expenditures, and other technological variables such as royalty and technical fee payments to the domestic and abroad, number of scientific and engineering personnels, etc., and $u_t \sim iid(0, \sigma^2)$. Z_{1t} is not observed, what we observe is P_t the total number of patents our firm has applied for up to period t. We further assume that

$$P_t = k \text{ if and only if } \alpha_k < Z_{1t} < \alpha_k + 1, \tag{47}$$

where $k = 0, 1, 2, ...m$ (a large positive number), $\alpha_0 = 0$, $\alpha_j \leq \alpha_{j+1}$, $j = 1, 2, ...m$, and $\alpha_{m+1} = \infty$.

If F is the distribution of u, then from (46) and (47), one gets

$$\text{Prob}\{ P_t = m \} = F(\alpha_{m-1} - \beta' X_t) - F(\alpha_m - \beta' X_t) \tag{48}$$

These models are known as ordered qualitative response models (see Amemiya [1985] and Maddala [1983] for more about these models).

We can use Logit or Probit specification to estimate (48), however, β in (48) could be estimated only up to a scale factor, namely we can estimate only β/σ.

7 Estimation and testing of the model

We illustrate estimation and testing problems for the rational expectations model giving rise to the system (30) and (41). It should be noted that by adding X-processes to the system (30) would not change the nature of the econometric problems, so we omit those for our expositional ease. (41) is not yet a regression equation as it does not involve an error term. We generate an error term in (41) as follows:

We view that ζ_t is a random process which is observed by the producer but not by the econometrician. Let us further assume that $c_4 = 0$ and that ζ_t follows a first order auto-regressive process, i.e., $\delta_4(L) = a_4$. So, the disturbance term in (41) is

$$\begin{aligned} e_t &= -\frac{b\lambda}{a_1\theta}\left(1-\lambda a_4\right)^{-1} a_{14}\zeta_t \\ &= -\frac{b\lambda}{a_1\theta}.q_{14}\left(1-\lambda a_4\right)^{-1}\left(1-a_4L\right)^{-1} w_{4t} \end{aligned}$$

i.e.,

$$(1-a_4L)e_t = -\frac{b\lambda}{a_1\theta}.q_{14}(1-\lambda a_4)w_{4t} \tag{49}$$

It is clear that the error term in (41) follows a first order auto-regressive process. For higher order auto-regressive processes, $\delta_4(L)$, it is straightforward to derive expressions similar to (49). Treating estimated Z_{1t} as observed technological knowledge, one can now use the method of maximum likelihood to estimate all the parameters.

Assuming that the model (30) is true for the Z_t- processes, the cross equation parameter restrictions could be used to test the hypothesis of rational expectations. Let L_1 be the likelihood of the sample of observations on R_t's when α's and g are unrestricted in (41). This involves estimating 9 parameters. Let L_2 be the likelihood of the sample on R_t's when α's and g are estimated as function of δ after plugging in the values of ν in (41). There are now 6 parameters to be estimated. Note that the Neyman - Pearson's likelihood ratio test criterion

$$-2(log L_2 - log L_1) \sim \chi^2_{9-6}$$

9-6 = number of restrictions under the hypothesis that the cross equation parameter restrictions are true.

In fact, the same test could also be used for testing the specifications of the model (30), under the assumption that the rational expectations hypothesis is true.

BIBLIOGRAPHY

Amemiya, T. [1985], ”*Advanced Theory of Econometrics*”, Cambridge: Harvard University Press.

Arrow, K. [1962], ”Economic Welfare and the Allocation of Resources for Inventions”, in (ed) R. R. Nelson, ”*The Rate and Direction of Inventive Activities*”, Princeton University Press, Princeton.

Bertsekas, D.P. [1976], ”*Dynamic Programming and Stochastic Control*”, Academic Press, New York.

Chow, Gregory C., [1975], ”*Analysis of Control of Dynamic Economic System”*, John Wiley & Sons, New York

Chow, Gregory C., [1981], *”Econometric Analysis by Control Methods”*, John Wiley & Sons, New York

Dasgupta, P. and J. Stiglitz [1980a], ”Industrial Structure and the Nature of Innovative Activity”, *Economic Journal* 90, 266-93.

——— [1980b], ”Uncertainty, Industrial Structure, the Speed of R&D”, em Bell Journal of Economics, 11, 1-28.

Evenson, R. and Y. Kislev [1976], ”A Stochastic Model of Applied R and D”, *Journal of Political Economy*, 84, 265-81.

Griliches, Zvi (ed) [1984], ”*R&D, Patents, and Productivity”*, The University of Chicago Press, Chicago.

——— [1979], Issues in Assessing the Contribution of Research and Development to Productivity Growth”, *The Bell J. Econ.*, 10(1), Spring.

Hansen Lars Peter and Thomas Sargent [1981], ”Formulating and Estimating Dynamic Linear Rational Expectations Models”, in Lucas, Robert and Thomas Sargent (eds), ”*Rational Expectations and Econometric Practice”*, Minneapolis:University of Minnesota Press.

Kamien, M. I. and N. L. Schwartz [1981], *”Market Structure and Innovation”*, Cambridge University Press, Cambridge.

Landes, David S. [1969], *”Unbounded Promethus: Technological Change and Industrial Development in Western Europe from 1750 to the present”*, Cambridge University Press, London.

Levin, R. C. [1981], ”Toward an Empirical Model of Schumpeterian Competition”, Working Paper Series A, no. 43, Yale School of Organization and Management.

Levin, R. C. and P. C. Reiss [1984], ”Tests of a Schumpeterian Model of R & D Market Structure”, in (ed) Zvi Griliches [1984].

Lucas, Robert [1976], ”Econometric Policy Evaluation: A Critique”, in K. Brunner and A.H. Meltzer (eds), *”The Phillips Curve and Labor Markets”*, Carneigie-Rochester Conferences on Public Policy 1, Amsterdam: North- Holland.

Maddala, G. S. [1983], *”Limited-Dependent and Qualitative Variables in Econometrics”*, Cambridge University Press, Cambridge.

Mairesse, Jaques and Mohamed Sassenou [1991], ”R&D and Productivity Growth: A survey of econometric studies at the firm level”, NBER Working Paper #3666.

Mohnen, P. [1992], ”R&D and Productivity Growth: a survey of the literature”, appeared as a chapter in, ”The relationship between R&D and Productivity Growth in Canada and other industrialized countries”, Ministry of Supply and Services, Ottawa, 1992.

Nelson, R. R. [1959], ”The Simple Economics of Basic Scientific Research”, *J. of Political Economy,* 67, 297-306.

——— [1982], ”The Role of Knowledge in R&D Efficiency”, *Quarterly Journal of Economics,* 453-70.

Nelson, R. R. and S. Winter [1977], ”Simulation of Schumpeterian Competition”, *American Economic Review,* 67, 271-6.

——— and ——— [1978], ”Forces Generating and Limiting Concentration under Schumpeterian Competition”, *Bell J. of Economics,* 9, 524-548.

Pakes, Ariel [1984], ”Patents, R & D and the Stock Market Rate of Return: A Summary of Some Empirical Results”, in (ed) Zvi Griliches [1984].

Pakes, Ariel and Zvi Griliches [1984], ”Patents and R & D at the Firm Level: A First Look”, in (ed) Zvi Griliches [1984].

Raut, L.K. [1988], ”R&D Behavior of Indian Firms: A Stochastic Control Model”, *Indian Economic Review,* vol.23(2):207-229.

Raut, L.K. [1995], ”R&D Spillover and Productivity Growth: Evidence from Indian Private Firms”, *Journal of Development Economics*, Vol.48:1-23.

Rosenberg, J. B. [1976], ”Research and Market Share: A Reappraisal of the Schumpeter Hypothesis”, *J. Industrial Economics*, XXV, 110-112.

Schmookler, J. [1966], *”Invention and Economic Growth”*, Harvard University Press, Cambridge, Mass.

Schumpeter, J. A. [1934], *"The Theory of Economic Development"*, Harvard University Press, Cambridge, Mass.

——— [1950], *"Capitalism, Socialism, and Democracy"*, 3rd ed., Harper, NY.